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# Barycentric Coordinates

Francisco Javier García Capitán
I.E.S. Alvarez Cubero, Priego de Córdoba, Spain
e-mail: garciacapitan@gmail.com
web: http://garciacapitan.99on.com/

**Abstract.** We give a short introduction to barycentric coordinates.

**Keywords.** barycentric coordinates, triangle geometry.

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## 1. Barycentric coordinates with respect to a triangle.

1.1. **Homogeneous barycentric coordinates.** Let $ABC$ be a triangle and $u, v, w \in \mathbb{R}$ such that $u + v + w \neq 0$. For any point $O$, let $P$ be the point on the plane such that $(u+v+w)\overrightarrow{OP} = u\overrightarrow{OA} + v\overrightarrow{OB} + w\overrightarrow{OC}$. We can prove that $P$ does not depend on $O$. For, if $(u+v+w)\overrightarrow{O'P'} = u\overrightarrow{O'A} + v\overrightarrow{O'B} + w\overrightarrow{O'C}$, then

$$\begin{aligned}(u+v+w)(\overrightarrow{O'P'} - \overrightarrow{OP}) =& u(\overrightarrow{O'A} - \overrightarrow{OA}) + v(\overrightarrow{O'A} - \overrightarrow{OA}) + w(\overrightarrow{O'A} - \overrightarrow{OA}) = \\ =&(u+v+w)\overrightarrow{O'O},\end{aligned}$$

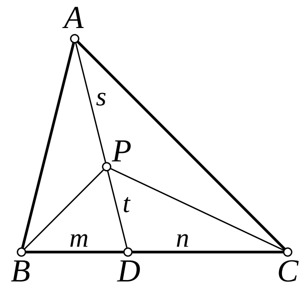
therefore $\overrightarrow{O'P'} = \overrightarrow{O'O} + \overrightarrow{OP} = \overrightarrow{O'P}$ and $P' = P$.

Hence we can define $P$ as the *center of mass* of the system formed by the points $A, B, C$ with weights $u, v, w$.

The barycentric coordinates of a point $P$ with respect to the triangle $ABC$ is a list $(x : y : z)$ of three numbers such that

$$x : y : z = (PBC) : (PCA) : (PAB).$$

Now we prove that $P$ is the center of mass of the system $A, B, C$ with weights $x, y, z$.



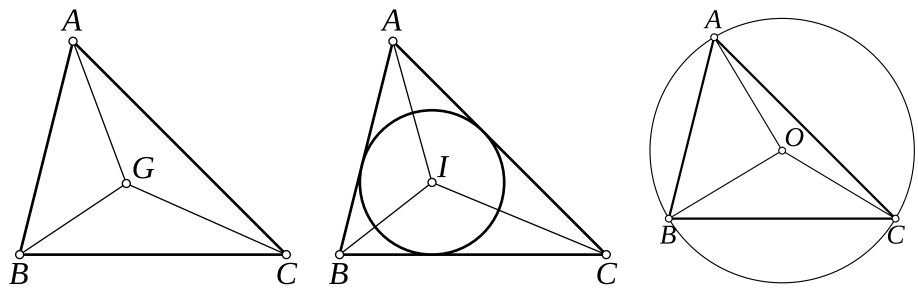

$$
\begin{aligned}
\overrightarrow{AD} =&\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AB} + \tfrac{m}{m+n}\overrightarrow{BC},\\
\overrightarrow{AP} =&\tfrac{s}{s+t}\overrightarrow{AD} = \tfrac{s}{s+t}\overrightarrow{AB} + \tfrac{sm}{(s+t)(m+n)}\overrightarrow{BC},\\
\overrightarrow{OP} - \overrightarrow{OA} =&\tfrac{s}{s+t}\left(\overrightarrow{OB} - \overrightarrow{OA}\right) + \tfrac{sm}{(s+t)(m+n)}\left(\overrightarrow{OC} - \overrightarrow{OB}\right),\\
\overrightarrow{OP} =&\tfrac{t}{s+t}\overrightarrow{OA} + \tfrac{sn}{(s+t)(m+n)}\overrightarrow{OB} + \tfrac{sm}{(s+t)(m+n)}\overrightarrow{OC}.
\end{aligned}
$$

Then $P$ is the center of mass of the system formed by $A, B, C$ with weights $t(m+n)$, $sn$ and $sm$. But,

$$\frac{sn}{sm} = \frac{DC}{BD} = \frac{(ADC)}{(ABD)} = \frac{(PDC)}{(PBD)} = \frac{(ADC) - (PDC)}{(ABD) - (PBD)} = \frac{\triangle(PCA)}{(PAB)}.$$

This proves that $y : z = (PCA) : (PAB)$. In the same way, we can prove that $x : y : z = (PBC) : (PCA) : (PAB)$.

*Examples.*




(1) The *centroid* $G$ has homogeneous barycentric coordinates $(1 : 1 : 1)$, because the areas $GBC$, $GCA$ and $GAB$ are equal to each other.
(2) The *incenter* $I$ has coordinates $a : b : c$, because the areas of the triangles $IBC$, $ICA$ e $IAB$ are respectively $\frac{1}{2}ar$, $\frac{1}{2}br$ and $\frac{1}{2}cr$, where $r$ is the inradius.
(3) The *circuncenter* $O$. If $R$ is the circumradius, the coordinates of $O$ are

$$
\begin{aligned}
&(OBC) : (OCA) : (OAB) =\\
=&\tfrac{1}{2}R^2\sin 2A : \tfrac{1}{2}R^2\sin 2B : \tfrac{1}{2}R^2\sin 2C =\\
=&\sin A\cos A : \sin B\cos B : \sin C\cos C =\\
=&a\cdot\frac{b^2+c^2-a^2}{2bc} = b\cdot\frac{c^2+a^2-b^2}{2ac} = c\cdot\frac{a^2+b^2-c^2}{2ab} =\\
=&a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2).
\end{aligned}
$$

(4) The points on line $BC$ have coordinates of the form $(0 : y : z)$. In the same way, points on lines $CA$ and $AB$ have coordinates $(x : 0 : z)$ and $(x : y : 0)$, respectively.

*Exercises.*

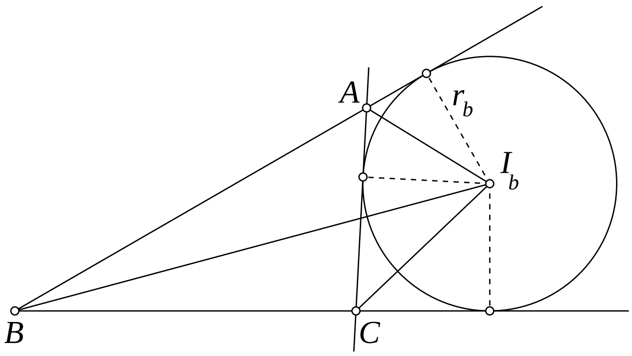
(1) Show that the sum of coordinates of the circumcenter equals to $4S^2$, where $S$ is twice the area of $ABC$.

We consider the Heron formula for the area of the triangle and do some algebraic manipulation:

$$\begin{aligned}
&a^2(b^2+c^2-a^2)+b^2(c^2+a^2-b^2)+c^2(a^2+b^2-c^2)=\\
=&2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4=\\
=&(a+b+c)(-a+b+c)(a-b+c)(a+b-c)=\\
=&4\cdot 4\cdot\frac{a+b+c}{2}\cdot\frac{-a+b+c}{2}\cdot\frac{a-b+c}{2}\cdot\frac{a+b-c}{2}=\\
=&4S^2.
\end{aligned}$$

(2) Find the coordinates of the excenters.

We consider the following figure that shows the triangle $ABC$ and the excenter $I_b$.




The barycentric coordinates of $I_b$ are

$$(I_bBC):(I_bCA):(I_bAB)=ar_b:-br_b:cr_b=a:-b:c.$$

We observe that the orientation of triangle $I_bCA$ is opposite the orientation of the other two, giving the negative second coordinate. In the same way, we have $I_a=(-a:b:c)$ and $I_c=a:b:-c$.

1.2. **Absolute barycentric coordinates.** Let $P$ be a point with (homogeneous barycentric) coordinates $(x:y:z)$. If $x+y+z\neq 0$, we get the *absolute* coordinates of $P$ by normalizing the coefficients so that their sum becomes 1:

$$P=\frac{x\cdot A+y\cdot B+z\cdot C}{x+y+z}.$$

If we have absolute coordinates of $P$ and $Q$, the point that divides the segment $PQ$ in the ratio $PX:XQ=p:q$ has absolute coordinates $\frac{qP+pQ}{p+q}$. However, since it is more convenient avoiding denominators in our calculations, we adapt the previous formula in the following way:

If $P=(u:v:w)$ and $Q=(u':v':w')$ are homogeneous barycentric coordinates satisfying $u+v+w=u'+v'+w'$, then the point $X$ that divides $PQ$ in the ratio $PX:XQ=p:q$ has homogeneous coordinates $(qu+pu':qv+pv':qw+pw')$.

*Exercises.*

(1) The orthocenter lies on Euler lines and divides $OG$ in the ratio $3:-2$. Prove that their barycentric coordinates can be written

$$H=(\tan A:\tan B:\tan C),$$

or equivalently,

$$H = \left(\frac{1}{b^2+c^2-a^2} : \frac{1}{c^2+a^2-b^2} : \frac{1}{a^2+b^2-c^2}\right).$$

We have seen that

$$\begin{aligned}
&O = \left(a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)\right),\\
&G = (1:1:1),
\end{aligned}$$

with sum $4S^2$ and 3 respectively. We first consider these coordinates with equal sum $12S^2$:

$$\begin{aligned}
&O = \left(3a^2(b^2+c^2-a^2) : 3b^2(c^2+a^2-b^2) : 3c^2(a^2+b^2-c^2)\right),\\
&G = (4S^2 : 4S^2 : 4S^2).
\end{aligned}$$

Now the first coordinate of $H$ is

$$\begin{aligned}
&(-2)3a^2(b^2+c^2-a^2) + 3\cdot 4S^2 =\\
=&-6(a^2b^2+a^2c^2-a^4) + 3(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4) =\\
=&3a^4-3b^4+6b^2c^2-3c^4 =\\
=&3(a^2-b^2+c^2)(a^2+b^2-c^2).
\end{aligned}$$

In the same way, the second and third coordinates are $3(-a^2+b^2+c^2)(a^2+b^2-c^2)$ and $3(-a^2+b^2+c^2)(a^2-b^2+c^2)$. If we divide them by

$$(-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)$$

we get

$$H = \left(\frac{1}{b^2+c^2-a^2} : \frac{1}{c^2+a^2-b^2} : \frac{1}{a^2+b^2-c^2}\right).$$

Now,

$$\frac{1}{b^2+c^2-a^2} = \frac{\frac{1}{2bc}}{\frac{b^2+c^2-a^2}{2bc}} = \frac{\frac{\sin A}{S}}{\cos A} = \frac{\tan A}{S},$$

threfore

$$H = (\tan A : \tan B : \tan C).$$

(2) Use that the nine point center $N$ divides $OG$ in the ratio $3 : -1$ to show that their barycentric coordinates can be written as

$$N = (a\cos(B-C) : b\cos(C-A) : c\cos(A-B)).$$

Starting from

$$\begin{aligned}
&O = \left(3a^2(b^2+c^2-a^2) : 3b^2(c^2+a^2-b^2) : 3c^2(a^2+b^2-c^2)\right),\\
&G = (4S^2 : 4S^2 : 4S^2),
\end{aligned}$$

the first coordinate of $N$ is

$$\begin{aligned}
&(-1)3a^2(b^2+c^2-a^2) + 3\cdot 4S^2 =\\
&= -3a^2b^2-3a^2c^2+3a^4+3(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)\\
&= 3(a^2b^2+a^2c^2+2b^2c^2-b^4-c^4).
\end{aligned}$$

To arrive the desired result, we use the formulas

$$\cos B = \frac{a^2+c^2-b^2}{2ac}, \quad S = ac\sin B$$

(and the corresponding ones for angle $C$). Then,

$$\begin{aligned}
\cos(B-C) = &\cos B\cos C + \sin B\sin C = \\
&= \frac{a^2+c^2-b^2}{2ac}\cdot\frac{a^2+b^2-c^2}{2ab} + \frac{S}{ac}\frac{S}{ab} = \\
&= \frac{(a^2+c^2-b^2)(a^2+b^2-c^2)+4S^2}{4a^2bc} = \\
&= \frac{2a^2b^2+2a^2c^2+4b^2c^2-2b^4-2c^4}{4a^2bc} = \\
&= \frac{1}{a}\cdot\frac{a^2b^2+a^2c^2+2b^2c^2-b^4-c^4}{2abc}.
\end{aligned}$$

Thus we have

$$N = (a\cos(B-C) : b\cos(C-A) : c\cos(A-B)).$$

## 2. Conway formula

2.1. **Notation.** If $\theta$ is any angle and $S$ is twice the area of $ABC$, we define $S_\theta = S\cot\theta$. In particular,

$$S_A = \frac{b^2+c^2-a^2}{2}, S_B = \frac{c^2+a^2-b^2}{2}, S_C = \frac{a^2+b^2-c^2}{2}.$$

Given two angles $\theta$ and $\phi$, we define $S_{\theta\phi} = S_\theta \cdot S_\phi$.

In this notation, we have the formulas :

(1) $S_B + S_C = a^2$, $S_C + S_A = b^2$, $S_A + S_B = c^2$.
(2) $S_{AB} + S_{BC} + S_{CA} = S^2$.

The first one is easy. The second one comes from the identity

$$\cot A\cot B + \cot B\cot C + \cot C\cot A = 1.$$

To prove this,

$$\begin{aligned}
&\cot A\cot B + \cot B\cot C + \cot C\cot A = \\
&= \cot A\,(\cot B + \cot C) + \cot B\cot C = \\
&= \frac{\cos A}{\sin A}\left(\frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}\right) + \frac{\cos B}{\sin B}\cdot\frac{\cos C}{\sin C} = \\
&= \frac{\cos A}{\sin A}\cdot\frac{\sin C\cos B + \sin B\cos C}{\sin B\sin C} + \frac{\cos B}{\sin B}\cdot\frac{\cos C}{\sin C} = \\
&= \frac{\cos A}{\sin A}\cdot\frac{\sin(B+C)}{\sin B\sin C} + \frac{\cos B}{\sin B}\cdot\frac{\cos C}{\sin C} = \\
&= \frac{\cos A}{\sin B\sin C} + \frac{\cos B\cos C}{\sin B\sin C} = \\
&= \frac{\cos B\cos C - \cos(B+C)}{\sin B\sin C} = \\
&= \frac{\sin B\sin C}{\sin B\sin C} = 1.
\end{aligned}$$

*Examples.*

(1) The orthocenter has coordinates

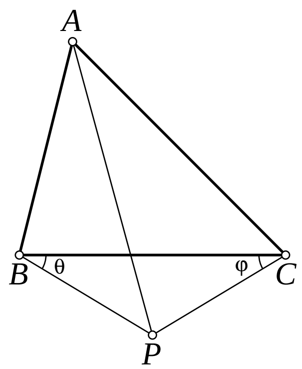
$$\left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right) = (S_{BC} : S_{CA} : S_{AB}).$$

(2) The circumcenter has coordinates

$$\left(a^2S_A : b^2S_B : c^2S_C\right) = (S_A(S_B + S_C) : S_B(S_C + S_A) : S_C(S_A + S_B)).$$

In this way, the sum of coordinates is $2(S_{AB} + S_{BC} + S_{CA}) = 2S^2$.

## 2.2. **Conway formula.**




Given a triangle $ABC$ and a point $P$, we consider the swing angles of $P$ with respect to $BC$ as $\theta = \angle CBP$ and $\varphi = \angle BCP$, considered in the range $-\frac{\pi}{2} \leqslant \theta, \varphi \leqslant \frac{\pi}{2}$. The angle $\theta$ is positive or negative according as angles $\angle CBP$ and $\angle CBA$ have different or the same orientation. For example, $\theta$ is taken positive if $\angle CBP$ and $\angle CBP$ have different or the same orientation.
The *Conway formula* gives the barycentric coordinates of a point from its swing angles $\theta$ and $\varphi$:

$$P = (-a^2 : S_C + S_\varphi : S_B + S_\theta).$$

To prove this, we use the sine theorem to triangle $PCB$,

$$\frac{BP}{\sin\varphi} = \frac{CP}{\sin\theta} = \frac{a}{\sin(\theta+\varphi)} \Rightarrow BP = \frac{a\sin\varphi}{\sin(\theta+\varphi)}, CP = \frac{a\sin\theta}{\sin(\theta+\varphi)}.$$

The area of the triangle $PBC$ is

$$(PBC) = \frac{1}{2}\cdot BC\cdot BP\cdot\sin\theta = \frac{a^2\sin\theta\sin\varphi}{2\sin(\theta+\varphi)}.$$

We can calculate the areas $(PCA)$ and $(PAB)$ in a similar way, giving:

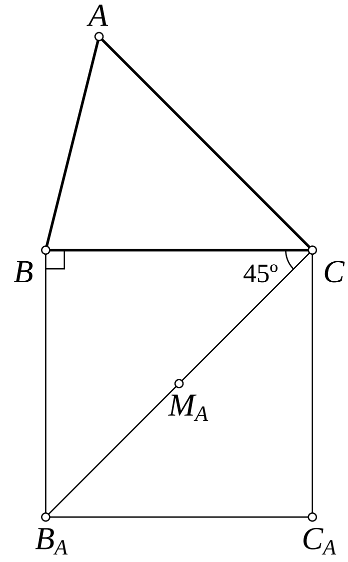
$$\begin{aligned}
&(PBC) : (PCA) : (PAB) = \\
=& -\frac{a^2\sin\theta\sin\varphi}{2\sin(\theta+\varphi)} : \frac{ba\sin\theta\sin(\varphi+C)}{2\sin(\theta+\varphi)} : \frac{ca\sin\varphi\sin(\theta+B)}{2\sin(\theta+\varphi)} = \\
=& -a^2 : \frac{ab\sin(\varphi+C)}{\sin\varphi} : \frac{ac\sin(\theta+B)}{\sin\theta} = \\
=& -a^2 : ab\cos C + ab\sin C\cot\varphi : ca\cos B + ac\sin B\cot\theta = \\
=& -a^2 : S_C + S_\varphi : S_B + S_\theta.
\end{aligned}$$

## 2.3. **Examples.**

### 2.3.1. *Square constructed on a side of the triangle.*

We consider the square $BCC_AB_A$ constructed externally on the side $BC$ of the triangle $ABC$. The point $B_A$ has swing angles $\theta = 90°$ and $\varphi = 45°$, therefore since $\cot 90° = 0$ and $\cot 45° = 1$, we have

$$\begin{aligned} B_A =& (-a^2 : S_C + S_\varphi : S_B + S_\theta) = \\ & (-a^2 : S_C + S : S_B). \end{aligned}$$




Similary,

$$C_A = (-a^2 : S_C : S_B + S).$$

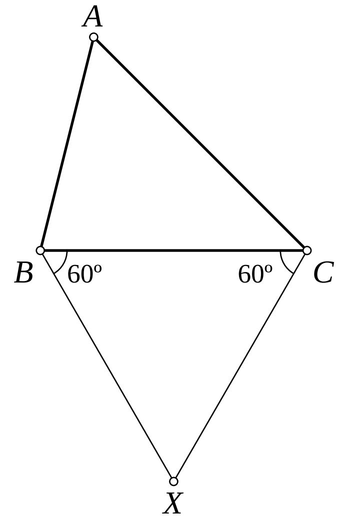
We caln calculate the midpoint of the square as the midpoint of $B_A$ and $C = (0 : 0 : S)$, giving

$$M_A = (-a^2 : S_C + S : S_B + S).$$

2.3.2. *Equilateral triangles constructed on a side.*

If we erect the equilateral triangle $BCX$ externally on $BC$, since $\cot 60° = 1/\sqrt{3}$, we have

$$X = \left(-a^2 : S_C + \frac{S}{\sqrt{3}} : S_B + \frac{S}{\sqrt{3}}\right).$$




Similarly, if we erect equilateral triangles $CYA$ and $AZB$ on $CA$ and $AB$,

$$Y = \left(S_C + \frac{S}{\sqrt{3}} : -b^2 : S_A + \frac{S}{\sqrt{3}}\right),$$

$$Z = \left(S_B + \frac{S}{\sqrt{3}} : S_A + \frac{S}{\sqrt{3}} : -c^2\right).$$

If we erect equilateral triangles $BX'C$, $CY'A$, $AZ'B$ internally, we get the points

$$X' = \left(-a^2 : S_C - \frac{S}{\sqrt{3}} : S_B - \frac{S}{\sqrt{3}}\right),$$

$$Y' = \left(S_C - \frac{S}{\sqrt{3}} : -b^2 : S_A - \frac{S}{\sqrt{3}}\right),$$

$$Z' = \left(S_B - \frac{S}{\sqrt{3}} : S_A - \frac{S}{\sqrt{3}} : -c^2\right).$$

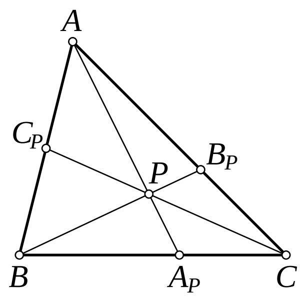
## 3. Cevians and traces

We call *cevians* of a point $P = (x : y : z)$ the lines joining $P$ with the vertices of the triangle.

The intersections $A_P, B_P, C_P$ of these cevians and the sides of the triangle are called the *traces* of $P$.

The coordinates of the traces are easily deduced from their geometrical meaning:

$$A_P = (0 : y : z),\ B_P = (x : 0 : z),\ C_P = (x : y : 0).$$



3.1. **Ceva theorem.** Three points $X, Y, Z$ on $BC$, $CA$ and $AB$ respectively are the traces of a point if and only if they have coordinates of the form $X = (0 : y : z)$, $Y = (x : 0 : z)$ and $Z = (x : y : 0)$ for some $x, y, z$.

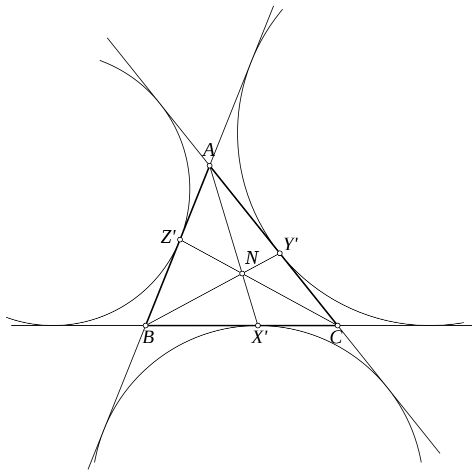
3.2. **Examples.**

3.2.1. *Gergonne point.* The points of tangency of the incircle with the sides are $X = (0 : s-c : s-b)$, $Y = (s-c : 0 : s-a)$ and $Z = (s-b : s-a : 0)$ that can be rewritten as $X = (0 : \frac{1}{s-b} : \frac{1}{s-c})$, $Y = (\frac{1}{s-a} : 0 : \frac{1}{s-c})$ and $Z = (\frac{1}{s-a} : \frac{1}{s-b} : 0)$. Therefore $AX$, $BY$ and $CZ$ intersect at the point with coordinates $(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c})$. This point is known as the *Gergonne point* $G_e$ of triangle $ABC$.

3.2.2. *Nagel point.* The points of tangency of the excircles and the corresponding sides have coordinates

$$X' = (0 : s-b : s-c),$$
$$Y' = (s-a : 0 : s-c),$$
$$Z' = (s-a : s-b : 0).$$

These are the traces of the point with coordinates $(s-a : s-b : s-c)$, known as *Nagel point* $N_a$ of triangle $ABC$.




3.2.3. *Exercise.* Show that the Nagel point $N_a$ lies on line $IG$ and $N_a$ divides $IG$ in the ratio $3 : -2$.

We consider $I = (3a : 3b : 3c)$ y $G = (2s : 2s : 2s)$ both with sum $6s$. Then,

$$-2I + 3G = (6s-6a : 6s-6b : 6s-6c) = (s-a : s-b : s-c) = N_a.$$

3.2.4. *Fermat Points.* We have calculated the coordinates of the points $X, Y, Z$ such that $BXC$, $CYA$ and $AZB$ are equilateral triangles constructed externally on the sides $BC$, $CA$, $AB$ of the triangle $ABC$.

These coordinates can be written as

$$X = \left( *** : \frac{1}{S_B + \frac{S}{\sqrt{3}}} : \frac{1}{S_C + \frac{S}{\sqrt{3}}} \right),$$
$$Y = \left( \frac{1}{S_A + \frac{S}{\sqrt{3}}} : *** : \frac{1}{S_C + \frac{S}{\sqrt{3}}} \right),$$
$$Z = \left( \frac{1}{S_A + \frac{S}{\sqrt{3}}} : \frac{1}{S_B + \frac{S}{\sqrt{3}}} : *** \right),$$

where the asterisks substitute quantities whose value is not needed.

The lines $AX$, $BY$, $CZ$ concur at the point

$$F_+ = \left( \frac{1}{S_A + \frac{S}{\sqrt{3}}} : \frac{1}{S_B + \frac{S}{\sqrt{3}}} : \frac{1}{S_C + \frac{S}{\sqrt{3}}} \right),$$

known as *first Fermat point.*

If we consider equilateral triangles constructed internally on the sides, we get the *second point of Fermat*:

$$F_{-} = \left( \frac{1}{S_A - \frac{S}{\sqrt{3}}} : \frac{1}{S_B - \frac{S}{\sqrt{3}}} : \frac{1}{S_C - \frac{S}{\sqrt{3}}} \right).$$

From

$$\begin{aligned}
S_A - \frac{S}{\sqrt{3}} =& S\left(\frac{\cos A}{\sin A} \pm \frac{1}{\sqrt{3}}\right) = \frac{S}{\sqrt{3}\sin A}\left(\sqrt{3}\cos A \pm \sin A\right) = \\
=& \frac{2S}{\sqrt{3}\sin A}\left(\frac{\sqrt{3}}{2}\cos A \pm \frac{1}{2}\sin A\right) = \\
=& \frac{2S}{\sqrt{3}\sin A}\left(\sin 60^\circ \cos A \pm \cos 60^\circ \sin A\right) = \\
=& \frac{2S}{\sqrt{3}} \frac{\sin(60^\circ \pm A)}{\sin A} = \\
=& \frac{4RS}{\sqrt{3}} \frac{\sin(60^\circ \pm A)}{a},
\end{aligned}$$

the coordinates of Fermat points can be written as follows:

$$\begin{aligned}
F_{+} &= \left( \frac{a}{\sin(60^\circ + A)} : \frac{b}{\sin(60^\circ + B)} : \frac{c}{\sin(60^\circ + C)} \right), \\
F_{-} &= \left( \frac{a}{\sin(60^\circ - A)} : \frac{b}{\sin(60^\circ - B)} : \frac{c}{\sin(60^\circ - C)} \right).
\end{aligned}$$

## 4. AREA OF A TRIANGLE

If $P = (x_1, y_1)$, $Q = (x_2, y_2)$ and $R = (x_3, y_3)$ are three points on the plane, we know that the area $(PQR)$ of the triangle $PQR$ is given by

$$(PQR) = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

If the barycentric coordinates of $P$, $Q$ and $R$ with respect to the triangle $ABC$ are $P = (u_1 : v_1 : w_1)$, $Q = (u_2 : v_2 : w_2)$ and $R = (u_3 : v_3 : w_3)$, then

$$\begin{aligned}
(u_1 + v_1 + w_1)P &= u_1 A + v_1 B + w_1 C, \\
(u_2 + v_2 + w_2)Q &= u_2 A + v_2 B + w_2 C, \\
(u_3 + v_3 + w_3)R &= u_3 A + v_3 B + w_3 C,
\end{aligned}$$

If we put $A = (r_1, s_1)$, $B = (r_2, s_2)$ and $C = (r_3, s_3)$, these equations can be written as

$$\begin{aligned}
(u_1 + v_1 + w_1)x_1 &= u_1r_1 + v_1r_2 + w_1r_3,\\
(u_1 + v_1 + w_1)y_1 &= u_1s_1 + v_1s_2 + w_1s_3,\\
(u_2 + v_2 + w_2)x_2 &= u_2r_1 + v_2r_2 + w_2r_3,\\
(u_2 + v_2 + w_2)y_2 &= u_2s_1 + v_2s_2 + w_2s_3,\\
(u_3 + v_3 + w_3)x_3 &= u_3r_1 + v_3r_2 + w_3r_3,\\
(u_3 + v_3 + w_3)y_3 &= u_3s_1 + v_3s_2 + w_3s_3.
\end{aligned}$$

and therefore

$$\begin{aligned}
&(u_1 + v_1 + w_1)(u_2 + v_2 + w_2)(u_3 + v_3 + w_3)(PQR) =\\
&= \frac{1}{2}\begin{vmatrix} u_1 + v_1 + w_1 & u_2 + v_2 + w_2 & u_3 + v_3 + w_3 \\ (u_1 + v_1 + w_1)x_1 & (u_2 + v_2 + w_2)x_2 & (u_3 + v_3 + w_3)x_3 \\ (u_1 + v_1 + w_1)x_1 & (u_2 + v_2 + w_2)y_2 & (u_3 + v_3 + w_3)y_3 \end{vmatrix} =\\
&= \frac{1}{2}\begin{vmatrix} u_1 + v_1 + w_1 & u_2 + v_2 + w_2 & u_3 + v_3 + w_3 \\ u_1r_1 + v_1r_2 + w_1r_3 & u_2r_1 + v_2r_2 + w_2r_3 & u_3r_1 + v_3r_2 + w_3r_3 \\ u_1s_1 + v_1s_2 + w_1s_3 & u_2s_1 + v_2s_2 + w_2s_3 & u_3s_1 + v_3s_2 + w_3s_3 \end{vmatrix} =\\
&= \frac{1}{2}\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}\begin{vmatrix} 1 & r_1 & s_1 \\ 1 & r_2 & s_2 \\ 1 & r_3 & s_3 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}(ABC).
\end{aligned}$$

When the homogeneous coordinates of $P$, $Q$, $R$ are normalized, we have

$$(PQR) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}(ABC).$$

## 5. Lines

5.1. **Equation of the line through two points.** We can use the formula given in the previous section for the area of the triangle to establish the equation of the line through two points $(u_1 : v_1 : w_1)$ and $(u_2 : v_2 : w_2)$:

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0.$$

5.1.1. *Examples.*

(1) The equations of the sides $BC$, $CA$ and $AB$ are, respectively, $x = 0$, $y = 0$ and $z = 0$. For example, since $B = (0 : 1 : 0)$ and $C = (0 : 0 : 1)$, the line $BC$ has equation

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = 0 \Leftrightarrow x = 0.$$

(2) The equation of the perpendicular bisector of $BC$ is $(b^2 - c^2)x + a^2(y - z) = 0$. For, this line goes through the midpoint of $BC$, with coordinates $(0 : 1 : 1)$ and the circumcenter $O$ of $ABC$ with coordinates

$$a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2).$$

Therefore the equation of the perpendicular bisector of $BC$ is

$$\begin{vmatrix} 0 & 1 & 1 \\ a^2(b^2+c^2-a^2) & b^2(a^2+c^2-b^2) & c^2(a^2+b^2-c^2) \\ x & y & z \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow (a^2c^2 - c^4 - a^2b^2 + b^4)x - a^2(b^2+c^2-a^2)(y-z) = 0 \Leftrightarrow$$
$$\Leftrightarrow (a^2c^2 - c^4 - a^2b^2 + b^4)x + a^2(b^2+c^2-a^2)(y-z) = 0 \Leftrightarrow$$
$$\Leftrightarrow (b^2+c^2-a^2)((b^2-c^2)x + a^2(y-z)) = 0 \Leftrightarrow$$
$$\Leftrightarrow (b^2-c^2)x + a^2(y-z) = 0.$$

(3) The internal bisector of angle $A$ joins the vertex $A = (1:0:0)$ and the incenter $I = (a:b:c)$. The equation is

$$\begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ x & y & z \end{vmatrix} = 0 \Rightarrow cy - bz = 0.$$

5.2. **Parallel lines.**

5.2.1. *Infinite points.* In order to get the equation of a parallel line we consider the infinite points. We know that each line has an infinite point and all infinite points lie on a line called line at infinity.

The line at infinity has equation $x+y+z=0$, since $x+y+z \neq 0$ always return an ordinary point.

The infinite point of the line $px+qy+rz=0$ is $(q-r : r-p : p-q)$, since their coordinates have sum zero and it lies on the line $px+qy+rz=0$.

On the other side if $P = (u_1 : v_1 : w_1)$ and $Q = (u_2 : v_2 : w_2)$ with $u_1+v_1+w_1 = u_2+v_2+w_2$, we can prove that the infinite point of line $PQ$ has coordinates $(u_1-v_1, u_2-v_2, u_3-v_3)$.

For example, since the orthocenter is $H = (S_{BC} : S_{CA} : S_{AB})$, the foot of the altitude from $A$ is $A_H = (0 : S_{CA} : S_{AB}) = (0 : S_C : S_B)$, with $S_C + S_B = a^2$ and the infinite point of the altitude through $A = (a^2 : 0 : 0)$ is $-a^2 : S_C : S_B)$.

5.2.2. *Parallel through a point.* The line that goes through $P = (u:v:w)$ parallel to $px+qy+rz=0$ has equation

$$\begin{vmatrix} q-r & r-p & p-q \\ u & v & w \\ x & y & z \end{vmatrix} = 0.$$

5.2.3. *Exercises.*

(1) Find the equations of the lines through $P = (u:v:w)$ parallel to the sides of the triangle.

The infinite point of $BC$ is, subtracting coordinates of $C$ from $B$, $(0,1,-1)$, then the line through $P$ parallel to $BC$ has equation

$$\begin{vmatrix} 0 & 1 & -1 \\ u & v & w \\ x & y & z \end{vmatrix} = 0 \Leftrightarrow (v+w)x - u(y+z) = 0.$$

The parallels to $CA$ and $AB$ are $(w+u)y - v(x+z) = 0$ and $(u+v)z - w(x+y) = 0$.

(2) Let $DEF$ be the medial triangle of $ABC$. Given a point $P$, let $XYZ$ be the cevian triangle with respect to $ABC$ and $UVW$ the medial triangle of $XYZ$. Find the point $P$ such that lines $DU$, $EV$ and $FW$ are parallel to the internal angle bisectors of angles $A$, $B$ and $C$ respectively.

We have :

$$A = (1:0:0), \quad B = (0:1:0), \quad C = (0:0:1),$$
$$D = (0:1:1), \quad E = (1:0:1), \quad F = (1:1:0),$$
$$X = (0:v:w), \quad Y = (u:0:w), \quad Z = (u:v:0).$$

Since

$$Y = (u:0:w) = ((u+v)u:0:(u+v)w),$$
$$Z = (u:v:0) = ((u+w)u:(u+w)v:0),$$

we have

$$U = ((2u+v+w)u:(u+w)v:(u+v)w).$$

If $DU$ is parallel to $AI$, both lines have the same point at infinity. Now since $2u^2+uv+uw+vw$ is the sum of coordinates of $U$, we consider

$$D = (0:u^2+uv+uw+vw:u^2+uv+uw+vw),$$

and by subtracting coordinates of $D$ and $U$ we get that the infinity point of line $DU$ is

$$(2u^2+uv+uw:-u^2-uw:-u^2-uv) = (2u+v+w:-u-w:-u-v).$$

The angle bisector $AI$ goes through $A = (a+b+c:0:0)$ and $I = (a:b:c)$, hence it infinite point is $(b+c:-b:-c)$.

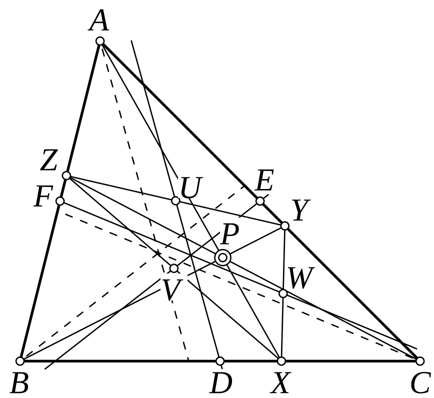
The two infinite points are the same when $u+w = kb$, $u+v = kc$ for some $k$. The same calculations for $EV$ and $FW$, give

$$\begin{cases} v+u = hc \\ v+w = ha \end{cases}, \quad \begin{cases} w+u = tb \\ w+v = ta \end{cases}$$

for some $h$ and $t$. This gives $k = h = t$ and $u, v, w$ are the solutions of the system of equations

$$\begin{cases} u+v = kc \\ u+w = kb \\ v+w = ka \end{cases},$$

that is , $u = k(b+c-a)$, $v = k(a+c-b)$, $w = k(a+b-c)$ or $P = (b+c-a:a+c-b:a+b-c)$ is the Nagel point of triangle $ABC$.



5.3. **Line intersection.** The intersection point of two lines

$$\begin{cases} p_1x + q_1y + r_1z = 0, \\ p_2x + q_2y + r_2z = 0 \end{cases}$$

is the point

$$\left( \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} : - \begin{vmatrix} p_1 & r_1 \\ p_2 & r_2 \end{vmatrix} : \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \right).$$

The infinite point of a line $l$ can be regarded as the intersection of $l$ and the line at infinity with equation $l_\infty : x + y + z = 0$.

Three lines $p_ix + q_iy + r_iz = 0$, $i = 1, 2, 3$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0.$$

5.3.1. *Examples.*

(1) Let $DEF$ be the medial triangle of $ABC$. Find the equation of the line $DI_a$, joining $D$ and the $A$-excenter. Show that the lines $DI_a$, $EI_b$ and $FI_c$ are concurrent and find their common point.

The equation of line $DI_a$ is

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 1 \\ -a & b & c \end{vmatrix} = 0 \Rightarrow (b-c)x + ay - az = 0.$$

Similarly we get

$$EI_b : -bx + (c-a)y + bz = 0.$$
$$FI_c : cx - cy + (a-b)z = 0.$$

The three lines concur if and only if the determinant of coefficients is zero:

$$\begin{vmatrix} b-c & a & -a \\ -b & c-a & b \\ c & -c & a-b \end{vmatrix} = \begin{vmatrix} -c & c & b-a \\ -b & c-a & b \\ c & -c & a-b \end{vmatrix} = 0.$$

To find the common point we solve the system formed by the two first equations:

$$\begin{aligned}(x:y:z) &= \left( \begin{vmatrix} a & -a \\ c-a & b \end{vmatrix} : - \begin{vmatrix} b-c & -a \\ -b & b \end{vmatrix} : \begin{vmatrix} b-c & a \\ -b & c-a \end{vmatrix} \right) = \\ &= (a(b+c-a) : b(a+c-b) : c(a+b-c)) = \\ &= (a(s-a) : b(s-b) : c(s-c)),\end{aligned}$$

known as *Mittenpunkt.*

(2) Let $DEF$ be the medial triangle of $ABC$ and $X, Y, Z$ the midpoints of the altitudes of $ABC$. Find the equations of lines $DX$, $EY$, $FZ$, show that they are concurrent and find their common point.

The orthocenter is $H = (S_{BC} : S_{CA} : S_{AB})$, hence the foot of the altitude from $A$ is $A_H = (0 : S_{CA} : S_{AB}) = (0 : S_C : S_B)$, with $S_C + S_B = a^2$.

Therefore, the midpoint of $A_H$ and $A = (a^2 : 0 : 0)$ is $X = (a^2 : S_C : S_B)$. The equation of line $DX$ is

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 1 \\ a^2 & S_C & S_B \end{vmatrix} = \begin{vmatrix} x & y & z-y \\ 0 & 1 & 0 \\ a^2 & S_C & S_B - S_C \end{vmatrix} = (S_B - S_C)x + a^2y - a^2z = 0.$$

Since $S_B - S_C = c^2 - b^2$, we get $DX : (c^2 - b^2)x + a^2y - a^2z = 0$. Similarly we get

$$EY : -b^2x + (a^2 - c^2)y + b^2z = 0,$$
$$FZ : c^2x - c^2y + (b^2 - a^2)z = 0.$$

Since

$$\begin{vmatrix} c^2 - b^2 & a^2 & -a^2 \\ -b^2 & a^2 - c^2 & b^2 \\ c^2 & -c^2 & b^2 - a^2 \end{vmatrix} = 0$$

because the first row is the sum of the other two, the three lines are concurrent.

To find the common point we solve:

$$\begin{aligned} &(x : y : z) = \\ &= \left( \begin{vmatrix} a^2 & -a^2 \\ a^2 - c^2 & b^2 \end{vmatrix} : - \begin{vmatrix} c^2 - b^2 & -a^2 \\ -b^2 & b^2 \end{vmatrix} : \begin{vmatrix} c^2 - b^2 & a^2 \\ -b^2 & a^2 - c^2 \end{vmatrix} \right) = \\ &= \left(a^2(a^2 + b^2 - c^2) : b^2(a^2 + b^2 - c^2) : c^2(a^2 + b^2 - c^2)\right) = \\ &=(a^2 : b^2 : c^2), \end{aligned}$$

and the three lines concur at the symmedian point of $ABC$.

(3) (*Vecten points*) We have already seen that the midpoint of the square $BCC_AB_A$ constructed externally on $BC$ is

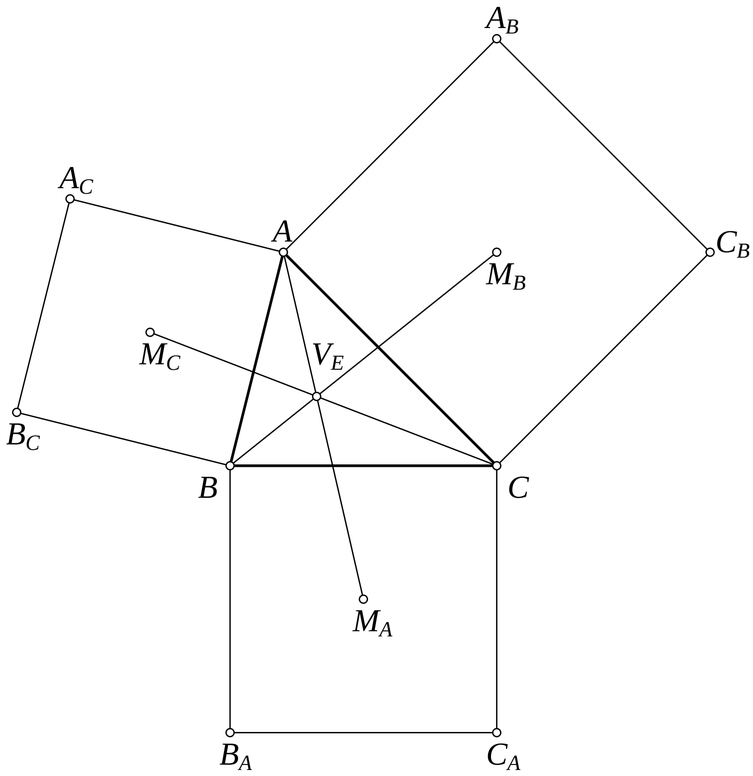
$$M_A = (-a^2 : S_C + S : S_B + S).$$

The line $AM_A$ has equation

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ -a^2 & S_C + S & S_B + S \end{vmatrix} = 0 \Rightarrow (S_B + S)y - (S_C + S)z = 0.$$

Similarly, we have

$$BM_B : (S_A + S)x - (S_C + S)z = 0$$
$$CM_C : (S_A + S)x - (S_B + S)y = 0$$




The three lines are concurrent:

$$\begin{vmatrix} 0 & S_B+S & -(S_C+S) \\ S_A+S & 0 & -(S_C+S) \\ S_A+S & -(S_B+S) & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} 0 & S_B+S & -(S_C+S) \\ 0 & S_B+S & -(S_C+S) \\ S_A+S & -(S_B+S) & 0 \end{vmatrix} = 0.$$

The common point is

$$V_E = ((S_B+S)(S_C+S) : (S_C+S)(S_A+S) : (S_A+S)(S_B+S)),$$

known as the *Vecten point* of $ABC$.

If we construct squares internally on the sides of the triangles, the we find the *inner Vecten point* of the triangle, whose coordinates are

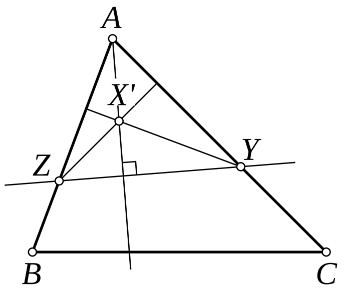
$$V_I = ((S_B-S)(S_C-S) : (S_C-S)(S_A-S) : (S_A-S)(S_B-S)).$$

5.4. **Perpendicular lines.** Given a line $\mathcal{L} : px+qy+rz=0$, it is interesting to calculate the point at infinity of all lines perpendicular to $\mathcal{L}$. The line $\mathcal{L}$ intersects $CA$ and $AB$ at the points $Y = (-r : 0 : p)$ and $Z = (q : -p : 0)$. We want to calculate the perpendicular line from $A$ to $\mathcal{L}$. First, we find the equations of perpendiculars from $Y$ to $AB$ and from $Z$ to $CA$. These are

$$\begin{vmatrix} S_B & S_A & -c^2 \\ -r & 0 & p \\ x & y & z \end{vmatrix} = 0, \qquad \begin{vmatrix} S_C & -b^2 & S_A \\ q & -p & 0 \\ x & y & z \end{vmatrix} = 0,$$

or

$$S_Apx + (c^2r - S_Bp)y + S_Arz = 0,$$
$$S_Apx + S_Aqy + (b^2q - S_Cp)z = 0.$$




The intersection point of these lines is the orthocenter of $AYZ$,

$$\begin{aligned} X' =&(***: S_Ap(S_Ar - b^2q + S_Cp) : S_Ap(S_Aq + S_Bp - c^2r)) = \\ =&(***: S_C(p-q) - S_A(q-r) : S_A(q-r) - S_B(r-p)), \end{aligned}$$

since $S_A + S_C = b^2$ and $S_A + S_B = c^2$.

Now the perpendicular from $A$ to $\mathcal{L}$ is the line $AX'$, whose equation is

$$\begin{vmatrix} 1 & 0 & 0 \\ *** & S_C(p-q) - S_A(q-r) & S_A(q-r) - S_B(r-p) \\ x & y & z \end{vmatrix} = 0,$$

or $-(S_A(q-r) - S_B(r-p))y + (S_C(p-q) - S_A(q-r))z = 0$.

Then, if we call $(f : g : h) = (q - r : r - p : p - q)$ to the infinite point of $\mathcal{L}$, the perpendicular to $\mathcal{L}$ from $A$ has equation

$$-(S_Af - S_Bg)y + (S_Ch - S_Af)z = 0,$$

with $(f' : g' : h') = (S_Bg - S_Ch : S_Ch - S_Af : S_Af - S_Bg)$ as infinite point, which will be also the infinite point of any line perpendicular to $\mathcal{L}$.

5.4.1. *Examples.*

(1) Show that the perpendicular to the sides through the points of tangency of the excircles are concurrent.

Let $X = (0 : s - b : s - c)$, $Y = (s - a : 0 : s - c)$ and $Z = (s - a : s - b : 0)$ the points of tangency of the excircles with the corresponding sides.

The infinite point of $BC$ is $(0 : 1 : 0) - (0 : 0 : 1) = (0 : 1 : -1)$. The infinite point of any perpendicular to $BC$ is

$$\begin{aligned} &(S_B \cdot 1 - S_C(-1) : S_C(-1) - S_A \cdot 0 : S_A \cdot 0 - S_B \cdot 1) = \\ =&(S_B + S_C : -S_C : -S_B) = (-a^2, S_C, S_B). \end{aligned}$$

and the perpendicular to $BC$ through $X$ has equation

$$\begin{vmatrix} 0 & s-b & s-c \\ -a^2 & S_C & S_B \\ x & y & z \end{vmatrix} = 0,$$

that is equivalent to $s(b - c)x + a(s - c)y - a(s - b)z = 0$.

If we calculate the perpendiculars to $CA$, $AB$ through $Y$, $Z$, we get :

$$\begin{aligned} -b(s-c)x + s(c-a)y + b(s-a)z = 0, \\ c(s-b)x - c(s-a)y + s(a-b)z = 0. \end{aligned}$$

The point of concurrence of these lines is known as the *Bevan point* of $ABC$.

## REFERENCES

[1] P.Yiu, *Introduction to the Geometry of the Triangle*, 2001, version of 2013, `http://math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry130411.pdf`.