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A Survey of Mathematics Discovered by Computers

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Abstract. This survey presents results in mathematics discovered by computers.

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1. Introduction

This survey presents some results in mathematics discovered by computers. We could not find (March 2015) in the literature new theorems in mathematics, discovered by computers, different from the results discovered by the "Discoverer". Hence, we present some results discovered by the computer program "Discoverer".

The computer program "Discoverer" is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science.

The creation of the computer program "Discoverer" began in June 2012. The "Discoverer" is created by the authors - Sava Grozdev and Deko Dekov. In 2006 was created the prototype of the "Discoverer", named the "Machine for Questions nd Answers". The prototype has discovered a few thousands new theorems. The prototype of the "Discoverer" has been created by using the *JavaScript*. The "Discoverer" is created by using *PHP* and *MySQL*.

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The reader may find description of the “Discoverer” in the paper [20].

The “Discoverer” produces theorems by using natural language. The reader could use the theorems without changes. The “Discoverer” is a module system. Each module is independent from the others. Each module can be independently improved and extended. Also, we could add easily new modules. Hence, the “Discoverer” could be easily improved and extended, and any further extension will make it better.

The “Discoverer” is in an early stage of development. We invite the interested persons and institutions to join the team and to work together with us on the “Discoverer”.

In this survey we present some results discovered by the “Discoverer”. The reader interested better to understand the “Discoverer” has to read the Grozdev-Dekov papers. The results included in this paper present a small part of the abilities of the “Discoverer”. We hope that the included results will give to the reader the possibility to feel the flower of the “Discoverer”. For complete description of the abilities of the “Discoverer” the reader has to consult the *Discoverer User Manual*.

The ideas of the “Discoverer” are effectively applicable in other areas of science, like biotechnology, medicine, biology, chemistry, physics, risk management, and so on. No doubt that the applications of the “Discoverer” in these areas will discover essential new results. Moreover, the “Discoverer” has the potential to extend a few times the current science.

In Grozdev-Dekov papers the reader often will hear about barycentric coordinates. The *homogeneous barycentric coordinates* of a point are denoted $(u : v : w)$. In the Grozdev-Dekov papers the same coordinates are named *barycentric coordinates* and are denoted by (u, v, w). In this journal the reader may see an introductory paper about the barycentric coordinates by Francisco Javier García Capitán [5]. For more information about barycentric coordinates we refer the reader to [5],[73],[74],[2],[65], [66],[48],[48]. See also [54], Glossary.

For some info about the Artificial Intelligence we refer the reader to [63],[64].

2. The Simon-Newell prediction

In 1958, in a seminal paper predicting future successes of artificial intelligence and operational research, Herbert Simon and Alan Newell [69] suggested that:

“Within ten years a digital computer will discover an important mathematical theorem”.

3. The Paulson Criterion

We need a criterion for possible realization of the Simon-Newell prediction. There are a few criteria. Below we consider the Paulson criterion.

The Paulson criterion is as follows [67]: In order a theorem in mathematics to be accepted as a theorem discovered by a computer, the following data has to be published in a mathematical journal:

(1) The statement of the theorem.
(2) A statement that the result is discovered by a computer.
(3) Direct or indirect statement that the theorem is new, not published before.

(4) The name of the computer program which has discovered the result.
(5) The name of the authors who have created the computer program.

The Strong Paulson criterion is as follows: The Paulson criterion plus

(6) The basic ideas of the computer program have to be published.
(7) If a number of theorems are published by the creators of the computer program, it should be clear that the theorems are not discovered by the people.

The computer program "Discoverer" satisfies the Strong Paulson criterion. Note that description of "Discoverer" is published in [20].

Simon and Newell speak about "important" theorem. The criterion about importance is not clear. We accept the following criterion: A new theorem is "important" if it generalizes published theorems, or it improves published theorems. Clearly, if a theorem solves a hypothesis, it also should be considered as "important", but we do not include this case, because we speak about theorems whose statements are not known in advance.

The computer program "Discoverer" satisfies the above criterion for "importance".

In March 2015 the predecessor of this journal, the "International Journal of Computer Generated Mathematics", announced inquiry about the realization of the Simon-Newell prediction. About 50 experts in the area of the AI have been invited to submit a theorem which satisfies the Paulson criterion. The result of the inquiry is as follows: The number of theorems which satisfy the Paulsen criterion is zero (except for the theorems discovered by the "Discoverer"). The inquiry proved that the only computer program which is a candidate for realization of the Simon-Newell prediction is the "Discoverer". The "Discoverer" realizes the Simon-Newell prediction. See [28] and the Grozdev-Dekov papers.

Note that currently there are more than 100 papers containing approximately 10 000 new theorems discovered by the "Discoverer" and its prototype. The current capacity of the "Discoverer" is approximately 500 000 new theorems. After the next extension of the "Discoverer", it is expected that it will be able to discover a few millions new theorems. But the value of the "Discoverer" is not in the number of discovered theorems. Simon and Newell speak about a separate theorem. The "Discoverer" discovers theories. See [14] and other Grozdev-Dekov papers.

4. The "Discoverer" improves the Steiner's Construction of the Malfatti Circles

The construction of the Malfatti circles is one of the famous mathematical problems. The problem was posed by the Italian geometer Gian Francesco Malfatti in 1803 [60]. A simple construction with compass and ruler of the Malfatti circles has been published by the great Swiss geometer Jacob Steiner in 1826. [72], Malfatti circles, [70]. As far as the authors know, the improvement of the Steiner's construction of the Malfatti circles, discovered by the "Discoverer" [25], is the first essential improvement of an important result in mathematics, discovered by a computer, and possibly, the first improvement of an important result in science, discovered by a computer.

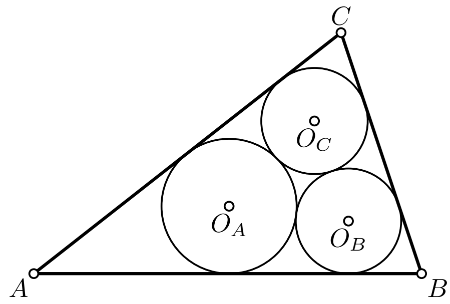

Figure 1

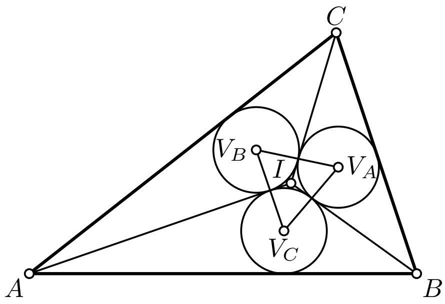

Figure 2

Given triangle ABC. The *Malfatti circles* are three circles inside a given triangle such that each circle is tangent to the other two and to two sides of the triangle. We denote by O_A, O_B and O_C the centers of the Malfatti circles inscribed in angles A, B and C, respectively. The triangle $O_AO_BO_C$ is known as the *Malfatti central triangle.* Figure 1 illustrates the Malfatti circles and the Malfatti central triangle.

Let I be the incenter of $\triangle ABC$ and let V_A, V_B and V_C be the incenters of the incircles c_1, c_2 and c_3, inscribed in triangles BCI, CAI and ABI, respectively. Recall that $\triangle V_AV_BV_C$ is known as the *de Villiers triangle.* Figure 2 illustrates the de Viliers triangle.

The mutual internal tangents to the circles c_1, c_2 and c_3 concur in a point S, known as the *Malfatti-Steiner Point.*

The Steiner's construction of the Malfatti circles has the following five stages:

Stage 1. Construct the internal angle bisectors and the incenter of $\triangle ABC$.

Stage 2. Construct the de Villiers triangle.

Stage 3. Construct the Malfatti-Steiner point S.

Stage 4. Construct the Malfatti central triangle.

Stage 5. Construct the Malfatti circles.

The computer program "Discoverer" has discovered a number of theorems related to the Malfatti circles. Below we give three of these theorems [7]:

Theorem 4.1. *The Malfatti-Steiner Point is the Isogonal Conjugate of the Incenter with respect to the de Villiers Triangle.*

Theorem 4.2. *The Malfatti-Steiner Point is the Second Kenmotu Point of the de Villiers Triangle.*

Theorem 4.3. *The Malfatti-Steiner Point is the Perspector of the Malfatti Central Triangle and the de Villiers Triangle.*

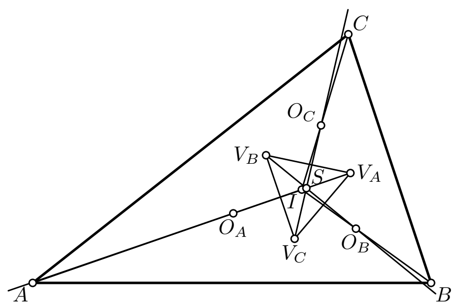

Figure 3

Theorems 4.1, 4.2 and 4.3 give alternative ways to the construction of the Malfatti-Steiner point. Figure 3 illustrates Theorem 4.3. The lines V_AO_A, V_BO_B and V_CO_C concur in point S.

Problem to the reader:

Problem 4.1. *Prove theorems 4.1, 4.2 and 4.3.*

In the paper [28] the reader may find two additional theorems giving two additional alternatives to the Steiner's construction.

The "Discoverer" improves the Steiner's Construction of the Malfatti circles. We use the theory of the complexity of the geometric constructions in order to obtain a numerical measure of the complexity of the solutions.

The first measure of the complexity of geometric constructions is proposed by Lemoine [72], Geometrography. In this paper we use the measure of Lazarov and Tabov [58], [70] which is summarized in Table 1.

In Table 1 LT means Lazarov-Tabov measure. The Lazarov-Tabov measure specifies the Lemoine's measure.

The explanation of row 1 in Table 1 is as follows. To place the edge of the ruler in coincidence with a point (Lemoine's operation R_1) – one point. To place the edge of the ruler in coincidence with a second point – one point. To draw a straight line (Lemoine's operation R_2) – one point. Hence, we obtain 3 points for drawing a straight line. The explanation of rows 2 and 3 in Table 1 is similar. Examples are given in [25].

	Construction	**LT**
1	Construct a line, which passes through two points.	3
2	Construct a circle with a given center and passing through another point.	3
3	Construct a circle with a given center and a radius, given by two points which are different form the center.	4
4	Construct a point, which is the intersection of two lines, circles, or a line and a circle.	1
5	Construct a point, which lies on a geometric figure or outside a geometric figure.	1

TABLE 1

The "Discoverer" improves Stage 4 as follows. Point O_A lies on the line AI. From theorem 2.3 point O_A lies also on the line V_AS. Hence, we can construct point O_A as the intersection point of lines AI and V_AS. In similar way we construct points O_B and O_C. Detailed calculations related to construction of the Malfatti circles are given in [25]. The Lazarov-Tabov measure of stage 4 of the Steiner's construction is 50, while the improved by "Discoverer" stage 4 has measure 12. The complexity of the improved by "Discoverer" stage 4 is 24% of the complexity of the Steiner's stage 4. Hence, the computer program "Discoverer" has discovered an essential improvement of stage 4 of the Steiner's construction of the Malfatti circles.

By using the "Discoverer" the reader may begin an extensive study of the Malfatti circles. In general, there are many interesting triads of circles in the Euclidean geometry which could be investigated by the "Discoverer".

5. Extremal Problems

In ([30], Problem 94, page 232) it is published the following problem, discovered by the 'Discoverer":

Problem 5.1. *Find a point P in the plane of a given triangle ABC, such that the sum*

$$\frac{|AP|^2}{b^2} + \frac{|BP|^2}{c^2} + \frac{|CP|^2}{a^2}$$

is minimal.

Answer: The First Brocard Point.

The problem is solved and generalized by Omran Kouba [55] and Moti Levy [61].

The "Discoverer" easily solves the famous extremal problems related to the Fermat Point and the Symmedian Point. The extremal problems engine of the "Discoverer" is under construction, but it is able to solve a number of extremal problems.

6. The Kiepert Hyperbola

In 1869 Ludwig Kiepert [53] introduced a hyperbola, now known as the Kiepert hyperbola. During the years a number of remarkable points of the triangle have been discovered to lie on the Kiepert hyperbola. In 1994 Eddy and Fritsch discovered that the Spieker Center and the Third Brocard Point lie on the Kiepert hyperbola ([3], Theorems 3 and 4). Eric Weisstein ([71], Kiepert Hyperbola), has published a list of 44 remarkable points, which lie on the Kiepert hyperbola. The reader may find more info about the Kiepert hyperbola e.g. in [59],[73].

The "Discoverer" has discovered the following theorem [13]:

Theorem 6.1. *Let P and Q are points, neither lying on a sideline of triangle ABC. If P and Q are isogonal conjugates with respect to ABC, then the Ceva product of their complements lies on the Kiepert hyperbola.*

Note that from the above theorem we can easily deduce theorem 3, [3].

The reader may find examples of points which lie on the Kiepert hyperbola in [13],[24]. These examples are discovered by the "Discoverer". In [11] the reader may find a list containing more than 2000 remarkable points which lie on the Kiepert hyperbola. The list is partially produced by using Theorem 6.1. The list is produced by the computer program "Discoverer".

Given a geometric figure like line, circle, conic, cubic and so on, the "Discoverer" easily discovers the remarkable points which lie on this geometric figure. In Grozdev-Dekov papers there are many examples. The points on geometric figures could be topics of investigation by using the "Discoverer".

7. The Steiner Circumellipse

Eric Weisstein ([71], Steiner circumellipse) has presented a list of 19 remarkable points, which lie on the Steiner circumellipse. Theorem 7.1 below, discovered by the "Discoverer", presents an unexpected result, which shows that we can obtain a new point on the Steiner circumellipse, if we start from an arbitrary point which lies on the Kiepert hyperbola. See [22].

Theorem 7.1. *The product of the Steiner point and any Kiepert perspector lies on the Steiner circumellipse.*

The Table 2, discovered by the "Discoverer", gives a few examples of the above theorem:

	Product of the Steiner point and the	Kimberling notation of the product
1	Centroid	X(99) = Steiner Point
2	Orthocenter	X(648)
3	Spieker center	X(190)
4	Third Brocard Point	X(670)
5	Tarry point	X(2966)

Table 2

Theorem 7.1 could be used as follows. We take the points which lie on the Kiepert hyperbola and form their products with the Steiner point. By this way we obtain a number of new points which lie on the Steiner circumellipse. In [12] the reader may find a list of more than 2000 notable points which lie on the Steiner circumellipse. The list is partially produced by using Theorem 7.1. The list is produced by the computer program "Discoverer".

8. Perspectors

In [71], [54] and other sources there are a number of theorems about perspective triangles and their perspectors. The "Discoverer" easily finds triangles which are perspective to a given triangle. In the supplementary material to the paper [14] the reader may find a list containing 183 of the main perspectors. The list extends the corresponding results given in the Kimberling's ETC [54].

Note that currently the "Discoverer" could easily discover more than 10 000 interesting perspectors. It is expected that the new extended version of the "Discoverer" will be able to discover more than one million new perspectors.

One example:

Theorem 8.1. *The Symmedian Point is the Perspector of Triangle ABC and the Medial Triangle of the Orthic Triangle.*

This is a theorem from [54], article X(6) The Symmedian Point. The result, as well as many similar results, is available in the 2006 edition of the "Computer-Generated Encyclopedia of Euclidean Geometry" (Contents > Theorems > Constructions > Perspector), which is produced by the prototype of the "Discoverer".

9. Geometric Constructions

The geometric constructions by straightedge and compass are one of the classical topics of the Euclidean geometry. In many cases new results in Euclidean geometry give us the possibility to find better (from the point of view of the geometrography) geometric construction. See for example the improvement of the Steiner's construction of the Malfatti circles, discovered by the "Discoverer" [25].

Note that we use the Lazarov-Tabov measure of compexity of geometric constructions. See [58],[70],[25].

A few Grozdev-Dekov papers contain results which allow the reader to find alternatives to interesting geometric constructions. One example:

Theorem 9.1. *The Yff Center of Congruence is the Internal Center of Similitude of the Incircle and the Circumcircle wrt the Pedal Triangle of the Incenter.*

This theorem gives an alternative to the existing constructions of the Yff Center of Congruence. See [9]. Note that results discovered by the prototype of the "Discoverer" related to the Yff Center of Congruence are noticed in the [72], Yff Center of Congruence.

Problem for the reader:

Problem 9.1. *Calculate the complexity of the variants of the geometric construction of the Yff Center of Congruence.*

Note that similar problem could be announced for many geometric objects.

10. Production of Problems

The "Discoverer" is an effective tool for production of problems for high school and university students. The methodology is given in [9] and in other papers. For many geometric objects the "Discoverer" is able to create new properties which could be reformulated as problems.

For example, the prototype of the "Discoverer" has discovered a number of properties about the Feuerbach Point. In 2008, 35 of the discovered properties about the Feuerbach Point have been published as problems in the predecessor of this journal, the IJCGM `http://www.ddekov.eu/j/contents.htm#2008`. By this moment there is no solved problem. We recommend the reader to solve a problem and to submit it for publications. Examples of problems from the list of the 35 problems:

Problem 1. *The Feuerbach Point lies on the Circle passing through the Symmedian Point, the Internal Center of Similitude of the Incircle and the Circumcircle and the Grinberg Point.*

Problem 18. *The Feuerbach Point is the Inverse of the Incenter in the Nine-Point Circle of the Fuhrmann Triangle.*

Problem 35. *The Feuerbach Point is the External Center of Similitude of the Circumcircle of the Johnson Triangle and the Circumcircle of the Outer Yff Triangle.*

11. Kosnita Products

The article [54], article X(54), Kosnita Point, contains a table with ten theorems about Kosnita Products. A table containing a few Kosnita products the reader may find also in Weisstein [71], Triangulation Point. The "Discoverer" has extended these tables. The extended table contains 39 theorems about Kosnita products. See table X-P given in the supplementary material of the paper [15].

For example, one of the theorems discovered by the "Discoverer" is as follows:

Theorem 11.1. *The Nine-Point Center is the Kosnita Product of the Orthocenter and the Centroid.*

The "Discoverer" has discovered the following general theorem about Kosnita products [9]:

Theorem 11.2. *The Kosnita Product of an arbitrary Point P and the Centroid is the Complement of the Complement of Point P.*

The reader may find information about Kosnita products in the paper [15].

The investigation of Kosnita products illustrates one of the abilities of the "Discoverer": If we take a group of a few similar theorems, the "Discoverer" can easily extend this group by adding to it new theorems.

12. CONICS

The conics are popular geometric object for investigations in the Analytic Geometry. This theme is interesting also for university professors and students.

The "Discoverer" has many abilities related to the conics. Currently additional abilities are under construction.

Below is an example about circumconics. Note that given a circumconic $\mathcal{C}$, its *associated circumconic* is the circumconic whose perspector is the center of $\mathcal{C}$. If two circumconics are associated, we say that their perspectors are *associated points.*

Table 3 below gives a few pairs of associated points, provided that the both points of the pair are available in the Kimberling's ETC [54]. Table 3 is produced by the "Discoverer". The reader may see results about conics in [23], [21].

	Point	Associated Point
1	Circumcenter	Symmedian Point
2	Orthocenter	X(1249)
3	Nine-Point Center	X(216)
4	Gergonne Point	X(3160)
5	Nagel Point	X(3161)
6	Spieker Center	Grinberg Point
7	Feuerbach Point	Center of the Stevanovic Circle
8	Brocard Midpoint	Symmedian Point of the Medial Triangle
9	Internal Center of Similitude of the Incircle and the Circumcircle	X(5452)
10	External Center of Similitude of the Incircle and the Circumcircle	X(478)

TABLE 3

13. Euler triangles, Euler products and Euler transforms

The classical Euler triangle is defined as follows: Let H be the orthocenter of $\triangle ABC$. Then the *Euler triangle* is the triangle whose vertices are the midpoints of segments AH, BH and CH.

Grozdev and Dekov [16] have generalized this definition as follows:

Definition 13.1. *Given an arbitrary point P in the plane of $\triangle ABC$. The Euler triangle of point P is the triangle whose vertices are the midpoints of segments AP, BP and CP.*

The vertices of the Euler triangle $E_1E_2E_3$ of point P have barycentric coordinates [16]:

$$E_1 = (2u + v + w, v, w),\ E_2 = (u, u + 2v + w, w),\ E_3 = (u, v, u + v + 2w).$$

The "Discoverer" has discovered the following theorems:

Theorem 13.1. *For any points P and Q, the Euler Triangle of Point P and the Triangle of Reflections of Point Q in the Vertices of Triangle ABC are perspective. The Perspector is the Point which divides internally the directed segment PQ in ratio 1 : 2.*

Theorem 13.2. *For any point P, the Euler Triangle of Point P and the Medial Triangle are perspective. The Perspector is the Complement of the Complement of Point P.*

Theorem 13.3. *For any point P, the Euler Triangle of Point P and the Half-Medial Triangle are perspective. The Perspector is the Complement of Point P.*

Theorem 13.4. *For any point P, the Euler Triangle of Point P is perspective with the following triangles: The Antimedial Triangle, the Inner Grebe Triangle, the Outer Grebe Triangle, the Johnson Triangle, the Inner Yff Triangle and the Outer Yff Triangle.*

Theorem 13.1 implies that to every two points P and Q is assigned a point. We call this point the *Euler product of points P and Q*. With the help of the "Discoverer" we can easily discover examples of Euler products of remarkable points. The supplementary material to [16] contains examples of Euler products. One of the theorems is as follows:

Theorem 13.5. *The Centroid is the Euler Product of the Spieker Center and the Incenter.*

Theorems 13.2 to 13.4 define eight transforms of points. We call these transforms the *Euler transforms*. The first of the Euler transforms assigns to point P the point which is the Perspector of the Medial triangle and the Euler triangle of P. In a similar way we define the other transforms. Note that the second transform coincides with the classical transform "complement of point P". In [16] are given a number of examples of these transforms. One of the theorems is as follows:

Theorem 13.6. *The Circumcenter is the Perspector of the Euler Triangle of the Orthocenter and the Half-Median Triangle.*

14. The Lester Circle

The Lester circle is a popular geometric object during the last years. See e.g. [75]. Any three points define a circle. The Lester circle is popular because four remarkable points lie on it. These are the Circumcenter, the Nine-Point Center, the Outer and Inner Fermat Points ([71], Lester Circle). In addition the point X(1117) lies on the Lester Circle. ([54], article X(1117)).

In many Grozdev and Dekov papers some theorems are presented as problems. The aim is the reader to prove the theorems and to publish the proofs. The "Discoverer" has discovered three new remarkable points which lie on the Lester circle. See [38], [39], [40]. Note that these new points are not available in the Kimberling's ETC [54].

Given a remarkable circle, the "Discoverer" easily finds the remarkable points which lie on it. We call "Lester circle" any remarkable circle which contains at least four remarkable points. The "Discoverer" has discovered many Lester circles (unpublished). The reader is invited to investigate the Lester circles by using the "Discoverer".

15. Orthogonal Circles

Weisstein ([71], Lester Circle), has noticed that the Lester circle is orthogonal to the Orthocentroidal circle.

In [41]-[47] the reader may find seven new remarkable circles which are orthogonal to the Lester circle. These new circles are discovered by the "Discoverer".

Given a remarkable circle, the "Discoverer" easily finds remarkable circles orthogonal on it.

16. The Haimov triangle

In *Mathematics and Informatics* (vol.56, 2013, no. 4), the reader may find the following problem whose author is Haim Haimov from Varna, Bulgaria:

Problem 16.1. *Given triangle ABC. The incircle of $\triangle ABC$ touches BC at A_1, CA at B_1 and AB at C_1. Denote by A_2 is the intersection point of the circumcircles of $\triangle ABC$ and $\triangle AB_1C_1$ which is different from point A. Analogously we define points B_2 and C_2. Prove that the lines A_1A_2, B_1B_2 and C_1C_2 concur in a point.*

The "Discoverer" has made a sample investigation of the above construction. We call triangle $A_2B_2C_2$ the *Haimov triangle* in honor of Haim Haimov [18]. The "Discoverer" has searched its database in order to identify the Haimov triangle. The result is as follows:

Theorem 16.1. *The Haimov triangle is the Circum-Anticevian Triangle of the Isogonal Conjugate of the Mittenpunkt*

The point of intersection of the lines A_1A_2, B_1B_2 and C_1C_2 is identified by the "Discoverer" as the External Center of Similitude of the Circumcircle and the Incircle:

Theorem 16.2. *The External Center of Similitude of the Circumcircle and the Incircle is the Perspector of the Haimov Triangle and the Intouch Triangle.*

The "Discoverer" has investigated the geometric constructions for the drawing of the Haimov triangle. In [18] two kinds of geometric constructions are presented. These geometric constructions are effective in many cases, so that we recommend the reader to read the paper [18]. The reader may find the use of these methods in a few Grozdev-Dekov papers.

17. Prasolov Products

The Prasolov point is a popular geometric object for investigation during the last years. The definition of the Prasolov point is generalized as follows:

Definition 17.1. *Given points U and P in the plane of $\triangle ABC$. Let $U_aU_bU_c$ be the cevian triangle of U. Denote by R_a, R_b and R_c the reflections of U_a, U_b and U_c in P, respectively. If the lines AR_a, BR_b and CR_c concur in a point, we say that the Prasolov product of U and P is defined. In this case the intersection point of the lines is the Prasolov product of U and P.*

Note that if U is the orthocenter of $\triangle ABC$ and P is the nine-point center of $\triangle ABC$, then the Prasolov product coincides with the classical Prasolov point.

The reader may find the following problem in [31]:

Problem 17.1. *Prove that the Prasolov product is defined, provided U is the Nagel point of $\triangle ABC$ and P is the Spieker center of $\triangle ABC$.*

The solution is given by Kouba in [56]. A solution may be found also in [32].

The reader may find similar generalizations of the Prasolov product in [33], [34], [35]. In these papers are given examples of analogs of the Prasolov product, discovered by the "Discoverer". For examples of Prasolov products, see the Table 4 below in which aaP denotes the anticomplement of the anticomplement of point P:

	U	P	Prasolov product
1	Orthocenter, X(4)	Taylor center, X(389)	X(3)
2	Symmedian point, X(6)	Brocard midpoint X(39)	X(76)
3	Nagel point, X(8)	Mittenpunkt, X(9)	X(7)
4	Kosnita point, X(54)	Taylor center, X(389)	X(52)
5	Centroid, X(2)	Arbitrary point P	aaP

Table 4

Note that an extensive investigation of Kosnita products could de made by using the "Discoverer".

18. Lalesco Products

Kimberling has included in his encyclopedia [54] seven theorems about Lalesco products. The paper [26] contains 1655 theorems about Lalesco products, discovered by the "Discoverer".

This is a typical situation. If we take a group of a few similar theorems, the "Discoverer" can easily extend this group with new theorems.

19. Corner Products

The Corner products are popular geometric objects for investigations during the last years. The "Discoverer" has found a number of theorems about the corner products [27], [37].

In [27] are given 1218 theorems about Cevian Corner Products. One example:

Theorem 19.1. *The Symmedian Point is the Cevian Corner Product of the Steiner Point and the Euler Reflection Point.*

This is a typical situation. If we take a group of a few similar theorems, the "Discoverer" can easily extend this group adding to it new theorems.

20. Scholarly Essays by Students

The aim of the "Discoverer" is to activate the interest of the students to mathematics and by this way to improve the high school and university education. The use of the "Discoverer" in education process we call "learning through discovery". The "learning through discovery" is a new important direction within the "learning through inquiry".

In a number of papers Grozdev and Dekov have discussed the methodology for writing a scholarly essay by a student. An example from the paper [29] is given below. In this special case the student investigates the reflections of remarkable points in the geometry of the triangle.

(1) The student chooses a set of remarkable points in the plane of $\triangle ABC$ (possibly from the database of the "Discoverer").
(2) The "Discoverer" produces a list of all reflections of these points. Note that the student could calculate these points by hand.
(3) "Discoverer" discovers which of the above produced points are not included in [54]. The help of "Discoverer" at this stage is essential.
(4) "Discoverer" discovers new theorems about the points which are not available in [54]. The help of "Discoverer" at this stage is essential.
(5) The student uses the computer program for dynamic geometry, like C.a.R. or GeoGebra, in order to investigate the ruler-and-compass constructions of the new points. The student produces macros for the new points and animations for the ruler-and-compass constructions. The student produces also computer graphics for his or her essay.
(6) The student uses computer algebra system, like Maple, in order to prepare the proofs of the theorems of the essay. The student calculates also the barycentric coordinates of the new points.
(7) The student prepares the scholarly essay. The essay contains as supplementary material the HTML-files, produced by "Discoverer", the C.a.R. files and graphics, and the Maple files, produced by the student.
(8) The new theorems, together with the barycentric coordinates of the new points could be submitted for publication in the Kimberling's encyclopedia [54] and in the Weisstein's encyclopedia [71]. The student has to submit

his essay for publication in the IJCDM, and in the Computer-Generated Encyclopedia of Euclidean Geometry [6].

During the work on the scholarly essay, the student will improve his skills to use a discovery system, like "Discoverer", a system for dynamic geometry, like C.a.R. (or GeoGebra, or Cabri), and a computer algebra system, like Maple. These skills are between the basic skills which the student has to master and improve during his education.

21. The Encyclopedia of Computer-Generated Euclidean Geometry

The "Discoverer" discovers theories. Hence, it is natural we to try to produce an encyclopedia containing the results of the "Discoverer". The answer to this idea is the on-line "Computer-Generated Encyclopedia of Euclidean Geometry" [6]. In 2006 the prototype of the "Discoverer" has produced a prototype of the encyclopedia. The new edition now is in preparation. See more about the encyclopedia in the paper [20].

The encyclopedia is a collection of results discovered by the "Discoverer" but we expect that it will be also a forum for discussions and results of students, teachers, professors and any persons interested in mathematics and in computer discoveries in mathematics.

We expect that the encyclopedia will contain about 100 000 new theorems discovered by the "Discoverer". It will contains also about twenty thousands known theorems.

We strongly encourage the interested persons to contribute to the encyclopedia. Especially are invited the high school and university students.

The encyclopedia contains the following sections:

Definitions: Definitions used in the encyclopedia.
Catalogue: The themes of the encyclopedia grouped in sections and subsections.
Articles: Articles about the "Discoverer" published in journals, conferences, and so on.
Constructions: Encyclopedia of geometric constructions based on the "Discoverer".
User's Manual: A manual for the users of the "Discoverer".
Student's Corner: Scholarly Essays by high school and university students, Bachelor's and Master's Thesis.
Teacher's Corner: Papers by school teachers and university professors.
Researcher's Corner: Papers by researchers.
Bulgarian Section: Papers in Bulgarian Language.
Russian Section: Papers in Russian Language.
Contributors: A list of contributors of the encyclopedia. Every contributor will have a separate web page containing description of his/her contributions.
Donate: We invite the interested persons to donate to the encyclopedia.

22. The Future

Euclid is said to have said to the first Ptolemy who inquired if there was a shorter way to learn geometry than the Elements: ...there is no royal road to geometry.

Now we have a royal road to geometry. It is not necessary we to be inventive. The "Discoverer" will tell us what is necessary. All which we have to do is to write our problem and to go to drink coffee. We will drink coffee and the "Discoverer" will work for us. It is easy.

23. Conclusion

The era of new knowledge discovered only by human being is over. We are going into a new era – the era of the computers-discoverers. The computers-discoverers are important. They could extend effectively the current science and technology.

It is time we to open our eyes and to see the advantages of the computers-discoverers. They work fast, they work hard, they work day and night, they do not need salaries, they do not do errors, and most important, they can do what the people cannot do.

Today "Discoverer" is the only computer program, able to discover new theorems in mathematics. We hope that tomorrow pleiads of computers-discoverers will help the people to move faster in science. Possibly, the computers-discoverers will discover new cheap energy which will change the world economics? But the people must be careful – the computers evolve much faster than the people. Tomorrow the computers could want to take the place of the people.

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