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Computer Discovered Mathematics: Hexyl-Anticevian Triangles

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Abstract. We introduce the notion of a Hexyl-anticevian triangle of a point P. The known Hexyl triangle coincides with the Hexyl-anticevian triangle of the Incenter. In this paper we present theorems about the Hexyl-anticevian triangle of a point P, and we also consider the special cases when P is the Incenter or the Centroid. All theorems and problems in this paper are discovered by the computer program "Discoverer". Theorems about other special cases, e.g. when P is the Circumcenter, Orthocenter, Symmedian Point, and so on, could be easily discovered by the "Discoverer" upon request.

Keywords. hexyl triangle, hexyl-antimedial triangle, triangle geometry, remarkable point, computer-discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See [4].

In this paper, by using the "Discoverer", we investigate the hexyl-anticevian triangles. The paper contains selected theorems about hexyl-anticevian triangles. We expect that the majority of these theorems are new, discovered by a computer. Many of the proofs of theorem are not presented here. We recommend to the reader to find the proofs and to submit them for publication.

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Recall the definition of a hexyl triangle. See e.g. [13], Hexyl triangle: Given a triangle ABC and the anticevian triangle $PaPbPc$ of the Incenter of triangle ABC. Let Ka be the point in which the perpendicular to AB through Pb meets the perpendicular to AC through Pc. Similarly define Kb and Kc. Then triangle $KaKbKc$ is known as the Hexyl Triangle.

We generalize the definition of the hexyl triangle as follows:

Given a triangle ABC and the anticevian triangle $PaPbPc$ of a point P. Let Ka be the point in which the perpendicular to AB through Pb meets the perpendicular to AC through Pc. Similarly define Kb and Kc. Then $\triangle KaKbKc$ is the Hexyl-Anticevian Triangle of point P. If P is the Incenter, the Hexyl-Anticevian Triangle of the Incenter (or the Hexyl-Excentral Triangle) coincides with the ordinary Hexyl Triangle.

Note that the "Hexyl-Anticevian Triangle of P" is different form the "Hexyl Triangle of the Anticevian Triangle of P".

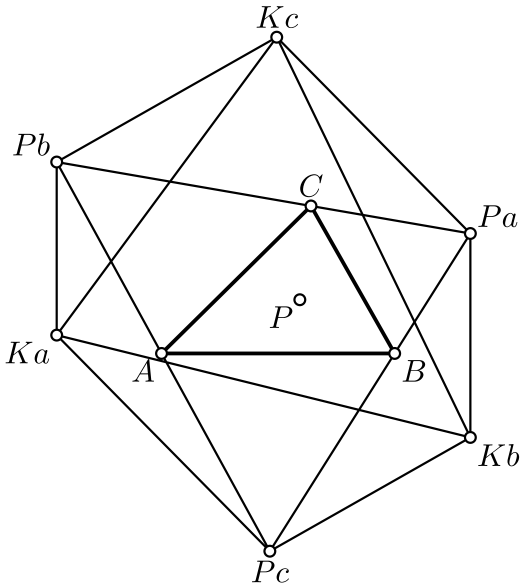

Figure 1.

Figure 1 illustrates the definition of the Hexyl-Anticevian Triangle of a point P. In figure 1, P is an arbitrary point, $PaPbPc$ is the anticevian triangle of P, $KaKbKc$ is the Hexyl-Anticevian triangle of P. Note that in this figure the hexagon $PaKcPbKaPcKb$ has parallel sides.

In this paper we present theorems about the Hexyl-Anticevian Triangle of a point P, and we consider the special cases when P is the Incenter or the Centroid. All theorems and problems in this paper are discovered by the computer program "Discoverer". Theorems about other special cases, e.g. when P is the Circumcenter, Orthocenter, Symmedian Point, and so on, could be easily discovered by the "Discoverer" upon request.

2. Preliminaries

In this section we review some basic facts about barycentric coordinates. We refer the reader to [15],[2],[1],[10],[6],[7],[9],[12]. The labeling of triangle centers follows Kimberling's ETC [8]. For example, X(1) denotes the Incenter, X(2) denotes the

Centroid, X(37) is the Grinberg Point, X(75) is the Moses point (Note that in the prototype of the "Discoverer" the Moses point is the X(35)), etc.

The reader may find definitions in [5], Contents, Definitions, and in [13],[14].

The reference triangle ABC has vertices $A = (1, 0, 0), B(0, 1, 0)$ and C(0,0,1). The side lengths of $\triangle ABC$ are denoted by $a = BC, b = CA$ and $c = AB$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is,

For all $k \in \mathbb{R}\diagdown\{0\} :\ P = (u, v, w)$ means that $P = (u, v, w) = (ku, kv, kw)$.

Given a point $P(u, v, w)$. Then P is *finite*, if $u + v + w \neq 0$. A finite point P is *normalized*, if $u + v + w = 1$. A finite point could be put in normalized form as follows: $P = (\frac{u}{s}, \frac{v}{s}, \frac{w}{s})$, where $s = u + v + w$. We use the Conway's notation:

$$S_A = \frac{b^2 + c^2 - a^2}{2},\ S_B = \frac{c^2 + a^2 - b^2}{2},\ S_C = \frac{a^2 + b^2 - c^2}{2},$$

The equation of the line joining two points with coordinates u_1, v_1, w_1 and u_2, v_2, w_2 is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0. \tag{2.1}$$

The intersection of two lines $L_1 :\ p_1x + q_1y + r_1z = 0$ and $L_2 :\ p_2x + q_2y + r_2z = 0$ is the point

$$(q_1r_2 - q_2r_1, r_1p_2 - r_2p_1, p_1q_2 - p_2q_1) \tag{2.2}$$

The infinite point of a line $L :\ px + qy + rz = 0$ is the point (f, g, h), where $f = q - r$, $g = r - p$ and $h = p - q$.

The equation of the line through point $P(u, v, w)$ and parallel to the line $L :\ px + qy + rz = 0$ is as follows:

$$\begin{vmatrix} f & g & h \\ u & v & w \\ x & y & z \end{vmatrix} = 0$$

The equation of the line through point $P(u, v, w)$ and perpendicular to the line $L :\ px + qy + rz = 0$ is as follows (The method discovered by Floor van Lamoen):

$$\begin{vmatrix} F & G & H \\ u & v & w \\ x & y & z \end{vmatrix} = 0 \tag{2.3}$$

where $F = S_Bg - S_Ch$, $G = S_Ch - S_Af$, and $H = S_Af - S_Bg$.

Three lines $p_ix + q_iy + r_iz = 0$, $i = 1, 2, 3$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0 \tag{2.4}$$

If the barycentric coordinates of points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$ are normalized, then the area of $\triangle P_1P_2P_3$ is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta. \tag{2.5}$$

where Δ is the area of the reference triangle ABC.

Given a point $P(u,v,w)$, the complement of P is the point $(v+w, w+u, u+v)$, the anticomplement of P is the point $(-u+v+w, -v+w+u, -w+u+v)$, the isotomic conjugate of P is the point (vw, wu, uv), and the isogonal conjugate of P is the point (a^2vw, b^2wu, c^2uv).

3. Hexyl-Anticevian Triangle of an Arbitrary Point P

Theorem 3.1. *Given a point P with barycentric coordinates P=(u,v,w). Then the barycentric coordinates of the Hexyl-Anticevian Triangle KaKbKc of P are as follows:*

$$\begin{gathered}
uKa = -u(3\,wb^4 - wc^4 - 2\,wc^2b^2 - 2\,wa^2b^2 - 2\,c^2a^2v - 2\,c^2b^2v \\
+2\,wc^2a^2 + 2\,a^2b^2v + 3\,c^4v - 2\,ua^2c^2 - 2\,uc^2b^2 - wa^4 \\
-2\,ua^2b^2 - a^4v - b^4v + ua^4 + ub^4 + uc^4), \\
vKa = 4\,wb^4u + 2\,wb^2c^2v + 2\,wb^2a^2v + 2\,wc^2a^2v + 4\,wc^2ub^2 - 4\,wa^2ub^2 \\
+2\,uc^2a^2v - 6\,uc^2b^2v + 2\,ua^2b^2v - uc^4v - wb^4v - wc^4v - wa^4v - ua^4v \\
-ub^4v - 2\,b^2v^2c^2 - 2\,a^2v^2b^2 - 2\,a^2v^2c^2 + a^4v^2 + b^4v^2 + c^4v^2, \\
wKa = -wb^4u - wc^4u + 2\,wb^2c^2v + 2\,wb^2a^2v + 2\,wc^2a^2v - 6\,wc^2ub^2 \\
+2\,wa^2ub^2 - 4\,uc^2a^2v + 4\,uc^2b^2v + 2\,wc^2ua^2 - 2\,w^2a^2b^2 + 4\,uc^4v - wb^4v \\
-wc^4v - wa^4v - wa^4u - 2\,w^2c^2b^2 - 2\,w^2a^2c^2 + w^2c^4 + w^2b^4 + w^2a^4, \\
uKb = wb^4u + wc^4u + 4\,wb^2a^2v - 4\,wc^2a^2v - 2\,wc^2ub^2 - 2\,wa^2ub^2 \\
+6\,uc^2a^2v - 2\,uc^2b^2v - 2\,wc^2ua^2 - 2\,ua^2b^2v + uc^4v + 2\,u^2a^2c^2 + 2\,u^2c^2b^2 \\
-4\,wa^4v + wa^4u + 2\,u^2a^2b^2 + ua^4v + ub^4v - u^2a^4 - u^2b^4 - u^2c^4, \\
vKb = -v(2\,a^2b^2v + 2\,c^2b^2v - a^4v - b^4v - c^4v + 2\,c^2a^2v + wc^4 \\
-2\,wc^2b^2 + 2\,wa^2b^2 + 2\,wc^2a^2 + 2\,ua^2c^2 + 2\,uc^2b^2 - 2\,ua^2b^2 \\
-3\,wa^4 + ua^4 + wb^4 - 3\,uc^4 + ub^4), \\
wKb = wb^4u + wc^4u - 2\,wb^2c^2v - 2\,wb^2a^2v + 6\,wc^2a^2v - 2\,wc^2ub^2 \\
-2\,wa^2ub^2 - 4\,uc^2a^2v + 4\,uc^2b^2v - 2\,wc^2ua^2 + 2\,w^2a^2b^2 - 4\,uc^4v + wb^4v \\
+wc^4v + wa^4v + wa^4u + 2\,w^2c^2b^2 + 2\,w^2a^2c^2 - w^2c^4 - w^2b^4 - w^2a^4, \\
uKc = wb^4u + wc^4u - 4\,wb^2a^2v + 4\,wc^2a^2v - 2\,wc^2ub^2 + 6\,wa^2ub^2 \\
-2\,uc^2a^2v - 2\,uc^2b^2v - 2\,wc^2ua^2 - 2\,ua^2b^2v + uc^4v + 2\,u^2a^2c^2 + 2\,u^2c^2b^2 \\
-4\,wa^4v + wa^4u + 2\,u^2a^2b^2 + ua^4v + ub^4v - u^2a^4 - u^2b^4 - u^2c^4, \\
vKc = -4\,wb^4u - 2\,wb^2c^2v + 6\,wb^2a^2v - 2\,wc^2a^2v + 4\,wc^2ub^2 - 4\,wa^2ub^2 \\
-2\,uc^2a^2v - 2\,uc^2b^2v - 2\,ua^2b^2v + uc^4v + wb^4v + wc^4v + wa^4v + ua^4v \\
+ub^4v + 2\,b^2v^2c^2 + 2\,a^2v^2b^2 + 2\,a^2v^2c^2 - a^4v^2 - b^4v^2 - c^4v^2, \\
wKc = -w(2\,wa^2b^2 + 2\,wc^2a^2 - wa^4 - wb^4 - wc^4 + 2\,wc^2b^2 - 3\,ub^4 \\
+uc^4 - 2\,c^2b^2v + 2\,a^2b^2v + 2\,c^2a^2v + 2\,uc^2b^2 \\
+2\,ua^2b^2 - 2\,ua^2c^2 + ua^4 + b^4v + c^4v - 3\,a^4v)
\end{gathered}$$

Theorem 3.2. *Let the vertices of the Hexyl-Anticevian Triangle of point $P(u,v,w)$ are finite points. Then the area $\Delta(P)$ of the Hexyl-Anticevian Triangle of P is as follows:*

$$\Delta(P) = \frac{4uvw\Delta}{(v+w-u)(w+u-v)(u+v-w)} \tag{3.1}$$

where Δ is the area of $\triangle ABC$.

Theorem 3.3. *Let $PaPbPc$ be the anticevian triangle of a point P and let $KaKbKc$ be the Hexyl-Anticevian Triangle of P. The hexagon $PaKcPbKaPcKb$ has parallel sides, that is, the lines $PaKb$ and $PbKa$ are parallel, the lines $PaKc$ and $PcKa$ are parallel, and the lines $PbKc$ and $PcKb$ are parallel.*

The hexagon $PaKcPbKaPcKb$ is drawn in Figure 1.

The locus of points whose cevian triangle is perspective to the Hexyl triangle is the cubic $K344$. The locus of the perspectors is the Darboux cubic. See [3], K004 and Table 6. Theorem 3.4 below gives an additional result:

Theorem 3.4. *The Hexyl-Anticevian Triangle of Point P and the Cevian Triangle of the Isotomic Conjugate of the Anticomplement of the Isogonal Conjugate of Point P are perspective.*

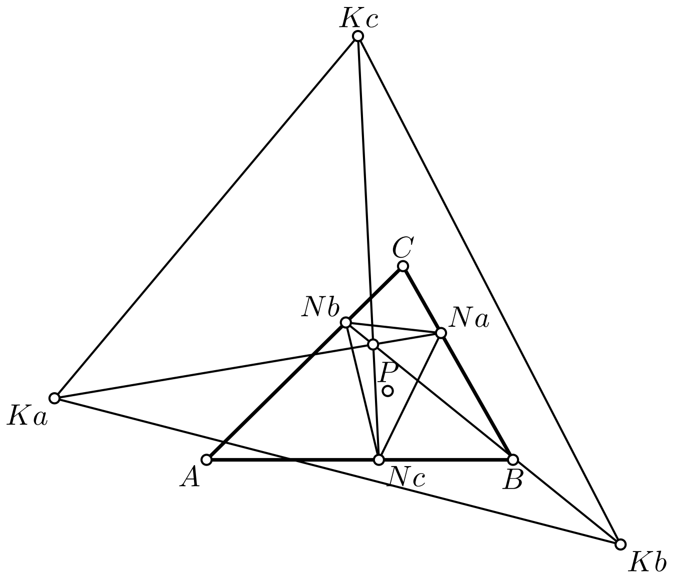

FIGURE 2.

In Figure 2, P is an arbitrary point, $NaNbNc$ is the Cevian triangle of the Isotomic Conjugate of the Anticomplement of the Isogonal Conjugate of P, and $KaKbKc$ is the Hexyl triangle. Then the lines $KaNa$, $KbNb$ and $KcNc$ concur in a point.

Problem 3.1. *Prove that through the vertices of the Anticevian triangle of P and the vertices of the Hexyl-Anticevian triangle of P is passing a conic. Find the center and the barycentric equation of this conic. If P is the Centroid, then the conic is a circle with center the Orthocenter.*

4. PROOFS OF THE THEOREMS

Proof of theorem 3.1. Let $P = (u, v, w)$. The vertices of the anticevian triangle $PaPbPc$ of P have coordinates $Pa = (-u, v, w)$, $Pb = (u, -v, w)$ and $Pc = (u, v, -w)$. By using (2.1) we find that the equation of the line AB is $z = 0$. By using (2.2) we find the equation of the line L_1 through point Pb and perpendicular to line AB as follows:

$$L_1 : \ (-wa^2 + wb^2 + wc^2 - 2\,c^2v)x + (wb^2 - wc^2 - wa^2 - 2\,uc^2)y \\ +(b^2v - c^2v - a^2v + ua^2 - ub^2 - uc^2)z = 0$$

Similarly, the equation of the line AC is $y = 0$, and the equation of the line L_2 through point Pc and perpendicular to line AC is as follows:

$$L_2: \; (-2\,wb^2 - a^2v + b^2v + c^2v)x + (wc^2 - wa^2 - wb^2 + ua^2 - ub^2 - uc^2)y$$
$$+(c^2v - a^2v - b^2v - 2\,ub^2)z = 0$$

By using (2.3) we find point Ka, the intersection of lines L_1 and L_2. The barycentric coordinates of point Ka are given in the statement of theorem 3.1. Similarly we find the barycentric coordinates of points Kb and Kc. □

Proof of theorem 3.2. By using (2.5) and the normalized barycentric coordinates of the Hexyl-Anticevian triangle of point P, given in the statement of theorem 3.1, we obtain the formula, given in the statement of the theorem 2. □

Proof of theorem 3.3. By using (2.1) and the barycentric coordinates of the Hexyl-Anticevian triangle of point P, given in the statement of theorem 3.1, we obtain the equation of the line $PaKb$ as follows:

$$PaKb: \; (wa^2 - wb^2 - wc^2 - 2\,c^2v)x + (-wb^2 + wc^2 + wa^2 - 2\,uc^2)y$$
$$+(b^2v - c^2v - a^2v + ua^2 - ub^2 - uc^2)z = 0.$$

The equation of the line L through point Pb and parallel to the line $PaKb$ is as follows:

$$L: \; (wa^2 - wb^2 - wc^2 + 2\,c^2v)x + (-wb^2 + wc^2 + wa^2 + 2\,uc^2)y$$
$$+(-b^2v + c^2v + a^2v - ua^2 + ub^2 + uc^2)z = 0.$$

Then we prove that point Ka lies on the line L. It follows that the lines $PaKb$ and $PbKa$ are parallel. Similarly we prove that the lines $PaKc$ and $PcKa$ are parallel, and the lines $PbKc$ and $PcKb$ are parallel.□

Proof of theorem 3.4. Let $P = (u, v, w)$. Denote by $Q = (uQ, vQ, wQ)$ the Isotomic Conjugate of the Anticomplement of the Isogonal Conjugate of Point P. Then

$$Q = \left(\frac{1}{-a^2vw + b^2wu + c^2uv}, \frac{1}{-b^2wu + c^2uv + a^2vw}, \frac{1}{-c^2uv + a^2vw + b^2wu} \right).$$

The vertices of the cevian triangle $QaQbQc$ of Q have barycentric coordinates $Qa = (0, vQ, wQ)$, $Qb = (uQ, 0, wQ)$ and $Qc = (uQ, vQ, 0)$. By using (2.1) and by using the barycentric coordinates of triangle $KaKbKc$, the Hexyl-Anticevian triangle of point P, we find the barycentric equations of lines $KaQa$, $KbQb$ and $KcQc$. Then by using (2.4), we prove that these three lines concur in a point. □

5. Special case: P = Incenter

This section contains selected theorems about the Hexyl triangle, discovered by the “Discoverer”. The theorems in this section extend the properties of the Hexyl Triangle. See [13], Hexyl Triangle.

It is known that the Hexyl Triangle is the Triangle of Reflections of the vertices of the Excentral Triangle in the Circumcenter. See e.g. [3], Table 6. Theorem 5.1 below gives a similar additional property of the Hexyl triangle.

Theorem 5.1. *The Hexyl Triangle is the Triangle of Reflections of the Bevan Point in the sidelines of the Excentral Triangle.*

Theorem 5.2. *The side lengths of the Hexyl triangle are as follows:*

$$\begin{aligned} a(I) &= |KbKc| = \frac{2a\sqrt{bc}}{\sqrt{(c+a-b)(a+b-c)}}, \\ b(I) &= |KcKa| = \frac{2b\sqrt{ca}}{\sqrt{(a+b-c)(b+c-a)}}, \\ c(I) &= |KaKb| = \frac{2c\sqrt{ab}}{\sqrt{(b+c-a)(c+a-b)}}. \end{aligned} \tag{5.1}$$

Theorem 5.3. *The side lengths in the previous theorem could be rewritten to*

$$a(I) = a \text{ csc}\left(\frac{A}{2}\right),\ b(I) = b \text{ csc}\left(\frac{B}{2}\right),\ c(I) = c \text{ csc}\left(\frac{C}{2}\right), \tag{5.2}$$

Theorem 5.4. *If $P = I$, then the formula (3.1) rewrites to the following formula, given in* [13]*, Hexyl Triangle:*

$$\Delta(I) = \frac{abc(a+b+c)}{4\Delta} \tag{5.3}$$

where Δ is the area of the reference triangle ABC.

Theorem 5.5. *The Hexyl triangle is congruent to the Excentral triangle.*

Problem 5.1. *From Theorem 5.5 and formulas (5.1) and (5.3) deduce the formulas (See Weisstein* [13]*, Excentral Triangle) for the side lengths and the area of the Excentral Triangle.*

Problem 5.2. *From Theorem 5.5 and the formulas for the side-lengths and the area of the Excentral triangle (See Weisstein* [13]*, Excentral Triangle) deduce the formulas (5.1) and (5.3)*

Theorem 5.6. *The Hexyl triangle is congruent to the Triangle of Reflections of the vertices of the Excentral Triangle in the Incenter.*

Theorem 5.7. *The Hexyl triangle and the Anticevian Triangle of the Mittenpunkt are not congruent, but the areas of these triangles are the same.*

Theorem 5.8. *The Hexyl triangle is similar to the Intouch Triangle.*

Theorem 5.9. *The Hexyl triangle is similar to the Fuhrmann Triangle.*

Theorem 5.10. *The Hexyl triangle is similar to the Yff Central Triangle.*

Theorem 5.11. *The Hexyl triangle is similar to the Triangle of Reflections of the Incenter in the Sidelines of Triangle ABC.*

Theorem 5.12. *The Hexyl triangle is similar to the Half-Cevian Triangle of the Nagel Point.*

It is known that the Centroid of the Hexyl Triangle is the point X(3576). See [8], article X(3576). Theorems 5.13-5.16 below give additional properties of this point.

Theorem 5.13. *The Centroid of the Hexyl Triangle is the Center of the Orthocentroidal Circle of the Excentral Triangle.*

Theorem 5.14. *The Centroid of the Hexyl Triangle is the Orthocenter of the Triangle of the Centroids of the Triangulation Triangles of the Incenter.*

Theorem 5.15. *The Centroid of the Hexyl Triangle is the Nagel Point of the Triangle of the Centroids of the Triangulation Triangles of the Bevan Point.*

Theorem 5.16. *The Centroid of the Hexyl Triangle is the Bevan Point of the Triangle of the Centroids of the Triangulation Triangles of the de Longchamps Point.*

Weisstein ([13], Hexyl Triangle) has presented a table which contains 12 notable points of the Hexyl triangle. Additional points of the Hexyl Triangle are noted in Kimberling [8] The theorems 5.17-5.18 below extend these results.

Theorem 5.17. *The Center of the Fuhrmann Circle of the Hexyl Triangle is the Incenter of the Excentral Triangle.*

Theorem 5.18. *The Center of the Orthocentroidal Circle of the Hexyl Triangle is the Centroid of the Excentral triangle.*

The Incenter of the Hexyl triangle is not available in Kimberling's ETC [8]. But this points have notable properties. Theorem 5.19 gives one of the properties of this point.

Theorem 5.19. *The Incenter of the Hexyl Triangle is the Center of the Fuhrmann Circle of the Excentral Triangle.*

The following two theorems present triangles which are perspective to the Hexyl triangle.

Theorem 5.20. *The Hexyl triangle is perspective with the Inner Yff Triangle. The perspector is not available in Kimberling's ETC* [8].

Theorem 5.21. *The Hexyl triangle is perspective with the Outer Yff Triangle. The perspector is not available in Kimberling's ETC* [8].

No remarkable points are known to lie on the Circumcircle of the Hexyl circle. See Weisstein [13], Hexyl Circle. Theorems 5.22-5.23 below describe remarkable points of the triangle which are not available in the Kimberling's ETC [8], but which lie on the Circumcircle of the Hexyl circle.

Theorem 5.22. *The Reflection of the Bevan Point in the Tarry Point lies on the Circumcircle of the Hexyl Triangle.*

Theorem 5.23. *The Reflection of the Bevan Point in the Steiner Point lies on the Circumcircle of the Hexyl Triangle.*

Theorem about circles:

Theorem 5.24. *The Nine-Point Circle of the Hexyl Triangle is the Circumcircle.*

6. SPECIAL CASE: P = CENTROID

Below we give selected theorems about the Hexyl-Anticevian triangle of the Centroid, (Hexyl-Antimedial triangle, for short) discovered by the "Discoverer".

Theorem 6.1. *The Hexyl-Antimedial triangle is congruent to the Antimedial triangle.*

Theorem 6.2. *The Hexyl-Antimedial Triangle is the Triangle of Reflections of the vertices of the Antimedial Triangle in the Orthocenter.*

Theorem 6.3. *The Hexyl-Antimedial Triangle is the Triangle of Reflections of the de Longchamps Point in the vertices of Triangle ABC.*

Theorem 6.4. *The Hexyl-Antimedial Triangle and the Euler Triangle are perspective and the perspector is the Centroid.*

Theorem 6.5. *The Hexyl-Antimedial Triangle and the Johnson Triangle are perspective and the perspector is the Orthocenter.*

Theorem 6.6. *The Circumcenter of the Hexyl-Antimedial Triangle is the Orthocenter.*

Theorem 6.7. *The Bevan Point of the Hexyl-Antimedial Triangle is the Nagel Point.*

Theorem 6.8. *The Tarry Point of the Hexyl-Antimedial Triangle is the Steiner Point of the Antimedial triangle.*

Theorem 6.9. *The Steiner Point of the Hexyl-Antimedial Triangle is the Tarryn Point of the Antimedial triangle.*

Theorem 6.10. *The Symmedian Point of the Hexyl-Antimedial Triangle is the reflection of the de Longchamps Point in the Symmedian Point.*

Theorem 6.11. *The Symmedian Point of the Hexyl-Antimedial Triangle is the reflection of the Retrocenter in the Orthocenter.*

Theorem 6.12. *The Symmedian Point of the Hexyl-Antimedial Triangle is the Retrocenter of the Triangle of the Orthocenters of the Anticevian Corner Triangles of the Orthocenter.*

Theorem 6.13. *The Symmedian Point of the Hexyl-Antimedial Triangle is the de Longchamps Point of the Triangle of the Orthocenters of the Antipedal Corner Triangles of the Retrocenter.*

Theorem 6.14. *The Gergonne Point of the Hexyl-Antimedial Triangle is the Anticomplement of the Anticomplement of the Midpoint of the Gergonne Point and the Orthocenter.*

Theorem 6.15. *The Mittenpunkt of the Hexyl-Antimedial Triangle is the reflection of the Gergonne Point in the Orthocenter.*

7. CONCLUSION

The computer program "Discoverer" fills a gap in the existing set of educational tools. It provides the possibility the students easily to discover new theorems in Euclidean geometry. By using the principles of "Discoverer", a number of similar computer programs could be created for the areas of high school teaching: physics, chemistry, biology, and so on. These computer programs could be considered as a tool for activation of the interest of the students. We call the use of "Discoverer" for educational purposes "learning through discovery". We may consider the "learning through discovery" as a new important direction within the "learning through inquiry".

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