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Posing and Solving Problems with Barycentric Coordinates

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Abstract. We use barycentric coordinates as a tool both for proposing new problems or solving them. Although barycentric coordinates are not always the most beautiful way to solve a problem, they may be a powerful tool to arrive quickly to a solution of the problem or to the creation of new ones.

Keywords. barycentric coordinates, problem solving, geometric constructions.

Mathematics Subject Classification (2010). 51-04.

1. Introduction

We use the standard notation in triangle geometry. See Yiu [1]. Denote by $a = BC$, $b = CA$ and $c = AB$ the sides of a triangle ABC. If we want to visualize all triangles ABC such that $f(a, b, c) = 0$ for some function f, we may fix B and C with cartesian coordinates $B = (-\frac{a}{2}, 0)$ and $C = (\frac{a}{2}, 0)$ and let $A = (x, y)$ vary on the plane meeting the conditions

$$b^2 = (x - \frac{a}{2})^2 + y^2, \quad c^2 = (x + \frac{a}{2})^2 + y^2 \tag{1}$$

In this way, if we eliminate b and c, we get an equation of a curve the form $\varphi(x, y, a) = 0$ as the locus of A.

As an example, if we consider the triangles such that $f(a, b, c) = b^2 + c^2 - a^2$, then we easily get $\varphi(x, y, a) = x^2 + y^2 - \frac{a^2}{4}$, and we arrive to the very well known fact that the locus for A is the circle with BC as diameter.

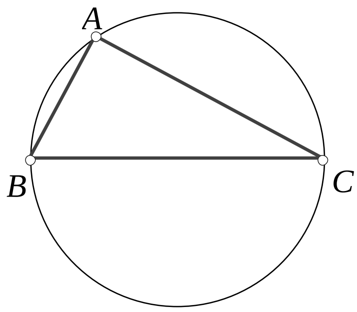

Figure 1

2. Some lines parallel to BC

In this section we look for triangles in which some particular lines are parallel to BC.

Problem 2.1. *Find all triangles ABC such that Euler line of ABC is parallel to line BC.*

Solution. The parallel to BC at $G = (1 : 1 : 1)$, containing the infinite point of BC, namely $(0 : -1 : 1)$ is the line $2x = y + z$. This line contains the orthocenter $H = (S_BS_C : S_CS_A : S_AS_B)$ if and only if

$$2S_BS_C = S_CS_A + S_AS_B = a^2S_A \Leftrightarrow a^2(b^2 + c^2 - a^2) = (c^2 + a^2 - b^2)(a^2 + b^2 - c^2).$$

Some further calculations using (1) lead to

$$12x^2 + 4y^2 = 3a^2 \Leftrightarrow \frac{4x^2}{a^2} + \frac{4y^2}{3a^2} = 1 \Leftrightarrow \frac{x^2}{\left(\frac{a}{2}\right)^2} + \frac{y^2}{\left(\frac{\sqrt{a}}{2}\right)^2} = 1.$$

This shows that the locus of A is an ellipse. The minor axis is the segment BC. The endpoints of the major axis are the vertices of the two equilateral triangles erected on the segment BC.

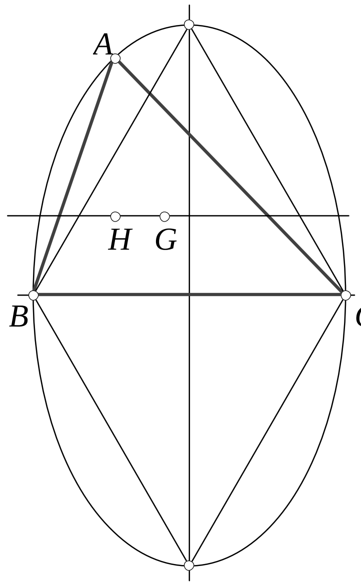

Figure 2

Problem 2.2. *Find all triangles ABC such that the Brocard axis of ABC is parallel to line BC.*

Solution. In this case, the line joining $(0 : -1 : 1)$ and $K = (a^2 : b^2 : c^2)$ has equation $(b^2 + c^2)x = a^2(y + z)$. This line goes through the circumcenter $O = (a^2 S_A : b^2 S_B : c^2 S_C)$ if and only if $(b^2 + c^2)a^2 S_A = a^2 (b^2 S_B + c^2 S_C)$, hence

$$\begin{aligned}(b^2+c^2)^2 - (b^2+c^2)a^2 =& b^2(c^2+a^2-b^2) + c^2(a^2+b^2-c^2)\\ =& (b^2+c^2)a^2 + b^2(c^2-b^2) + c^2(b^2-c^2)\\ =& (b^2+c^2)a^2 - (b^2-c^2)^2,\end{aligned}$$

and using (1) we get the equation

$$16y^4 + (32x^2 - 8a^2)y^2 + 16x^4 - 8a^2x^2 - 3a^4 = 0,$$

that using $a = 2m$ is equivalent to

$$y^4 + (2x^2 - 2m^2)y^2 + x^4 - 2m^2x^2 - 3m^4 = 0. \tag{2}$$

We can plot any of these curves. However, it is more interesting to find a ruler and compass constructions of these triangles ABC. If we solve (2) for y we find

$$y^2 = m^2 - x^2 \pm 2m\sqrt{m^2 - x^2} = \sqrt{m^2 - x^2}\left(\sqrt{m^2 - x^2} \pm 2m\right).$$

This gives a simple construction for points $A = (x, y)$ satisfying (2).

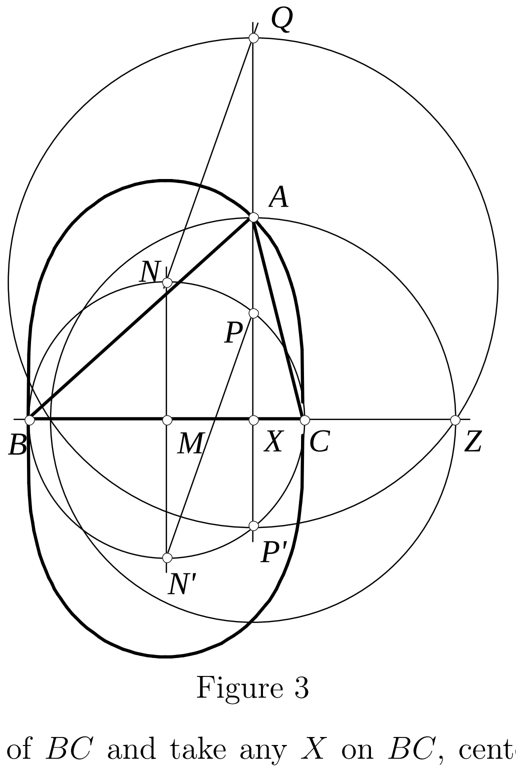

Figure 3

Call M the midpoint of BC and take any X on BC, center of the circle (BC) with BC as diameter. Let NN' the diameter of (BC) perpendicular to BC and PP' the chord through X perpendicular to BC. Let Q be the intersection of XP and the line through N parallel to $N'P$. Let Z be one of the intersection points of BC and the circle $(P'Q)$ with $P'Q$ as diameter. Let A any of the intersection points of XP and the circle centered at X with XZ as radius. Then the triangle ABC is a solution of our problem.

Figure 3 shows the locus of A, a closed symmetric curve respect to BC.

3. CONCURRENCES

Problem 3.1. *Let Y_a, Z_a be the contact points of the sides of the triangle ABC and the $A-$excircle, and define Z_b, X_b and X_c, Y_c cyclically. The triangle formed by the lines Y_aZ_a, Z_bX_b and X_cY_c is perspective with the medial triangle of ABC.*

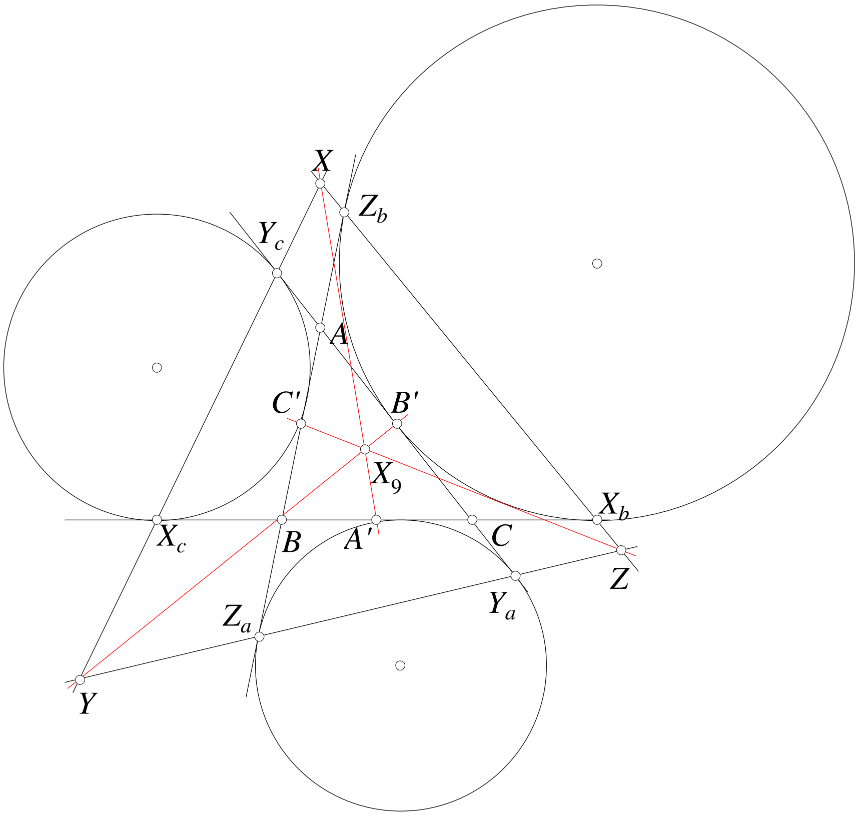

Figure 4

From $CY_a : Y_aA = -(s-b) : s$ and $AZ_a : Z_aB = s : -(s-c)$ we have $Y_a = (s-b : 0 : -s)$ and $Z_a = (s-c, -s, 0)$, and we can calculate the equation of the line $Y_aZ_a : sx + (s-c)y + (s-b)z = 0$. Similarly we have the lines $Z_bX_b : (s-c)x + sy + (s-a)z = 0$ and $X_cY_c : (s-b)x + (s-a)y + sz = 0$. These lines intersect at $X = (-a(b+c) : S_C : S_B)$. If $A'B'C'$ is the medial triangle of ABC, the line XA' has equation $(b-c)x + a(y-z) = 0$. In the same way we can calculate the lines $YB' : (c-a)y + b(z-x) = 0$ and $ZC' : (a-b)z + c(x-y) = 0$. The lines XA', YB' and CZ' concur at the point $X_9 = (a(s-a) : b(s-b) : c(s-c))$, known as the *Mittenpunkt* of ABC.

4. PROBLEM MATHEMATICAL REFLECTIONS O333

This is problem $O333$ of magazine Mathematical Reflections:

Problem 4.1. *Let ABC be a scalene acute triangle and denote by O, I, H its circumcenter, incenter, and orthocenter, respectively.Prove that if the circumcircle of triangle OIH passes through one of the vertices of triangle ABC then it also passes through one other vertex.*

Proposed by Josef Tkadlec, Charles University, Czech Republic

Solution. We consider an inversion with respect to the circumcircle. The circle HIO inverts on the line $X_{36}X_{186}$ joining X_{36}, the inverse of I and X_{186}, the inverse of H. We calculate later that this line has equation

$$\frac{(b-c)(s-a)S_A}{\cos A-\frac{1}{2}}x+\frac{(c-a)(s-b)S_B}{\cos B-\frac{1}{2}}y+\frac{(a-b)(s-c)S_C}{\cos C-\frac{1}{2}}z=0,$$

where we have used the usual notation

$$S_A=\frac{b^2+c^2-a^2}{2}, S_B=\frac{c^2+a^2-b^2}{2}, S_C=\frac{a^2+b^2-c^2}{2}, s=\frac{a+b+c}{2}.$$

The line $X_{36}X_{186}$ goes through A if and only if

$$(b-c)S_A\left(\cos B-\frac{1}{2}\right)\left(\cos C-\frac{1}{2}\right)=0.$$

Since ABC is a scalene acute triangle, we must have $B=60^\circ$ or $C=60^\circ$. Suppose, for example, that $B=60^\circ$. Then working backwards, the circle HIO also goes through C.

To complete the proof we calculate the barycentric coordinates of X_{36} and X_{180} and the equation of line $X_{36}X_{186}$.

Coordinates of X_{36}. Let I' be the inverse of I with respect to the circumcircle. From $OI\cdot OI'=R^2$ and $OI^2=R^2-2Rr$ (Euler formula), we get

$$\frac{OI'}{I'I}=\frac{OI'}{OI-OI'}=\frac{OI\cdot OI'}{OI^2-OI\cdot OI'}=\frac{R^2}{OI^2-R^2}=-\frac{R}{2r}=-\frac{abc}{4\Delta}:\frac{2\Delta}{s}=-\frac{abcs}{2S^2},$$

where $S=2\Delta$ is twice the area of the triangle ABC.

Now the points $O=(a^2S_A:b^2S_B:c^2S_C)$ and $I=(a:b:c)$ have sum of its coordinates $2S^2$ and $2s$ respectively, therefore sO and s^2I have equal sum (weight) and we can get algebraically $I'=2S^2(sO)-abcs(S^2I)=S^2s(2O-abcI)$. Hence the first coordinate of I' is

$$a^2(b^2+c^2-a^2)-a^2bc=a^2(b^2+c^2-a^2-bc)=2abc\cdot a\left(\cos A-\frac{1}{2}\right),$$

and the barycentric coordinates or X_{36} are

$$X_{36}=I'=\left(a\left(\cos A-\frac{1}{2}\right):b\left(\cos B-\frac{1}{2}\right):c\left(\cos C-\frac{1}{2}\right)\right).$$

Coordinates of X_{186}. Let H' be the inverse of H. From $OH\cdot OH'=R^2$ and the very well known formul $OH^2=R^2-8R^2\cos A\cos B\cos C$, we get

$$\frac{OH'}{H'H}=\frac{R^2}{OH^2-R^2}=\frac{R^2}{-8R^2\cos A\cos B\cos C}=\frac{-1}{8\cos A\cos B\cos C}=-\frac{a^2b^2c^2}{8S_AS_BS_C}.$$

The sum of the coordinates of $H=(S_BS_C:S_CS_A:S_AS_B)$ is S^2, therefore $H'=(8S_AS_BS_C)O-(a^2b^2c^2)(2H)$. The first coordinate of H' is

$$8S_AS_BS_C\cdot a^2S_A-a^2b^2c^2\cdot 2S_BS_C=2a^2S_BS_C\left(4S_A^2-b^2c^2\right)$$
$$=8a^2b^2c^2S_BS_C\left(\cos A+\frac{1}{2}\right)\left(\cos A-\frac{1}{2}\right),$$

and the barycentric coordinates of $H'=X_{186}$ are

$$X_{186}=\left(\frac{\cos A^2-\frac{1}{4}}{S_A}:\frac{\cos B^2-\frac{1}{4}}{S_B}:\frac{\cos C^2-\frac{1}{4}}{S_C}\right).$$

Equation of the line $X_{36}X_{186}$. The equation of $X_{36}X_{186}$ has the form $ux+vy+wz=0$, where the coefficient u is

$$\begin{vmatrix} b\left(\cos B-\frac{1}{2}\right) & c\left(\cos C-\frac{1}{2}\right) \\ \frac{\cos^2 B-\frac{1}{4}}{S_B} & \frac{\cos^2 C-\frac{1}{4}}{S_C} \end{vmatrix} = \frac{\left(\cos B-\frac{1}{2}\right)\left(\cos C-\frac{1}{2}\right)}{S_BS_C}\begin{vmatrix} bS_B & cS_C \\ \cos B+\frac{1}{2} & \cos C+\frac{1}{2} \end{vmatrix}$$
$$=-\frac{\left(\cos B-\frac{1}{2}\right)\left(\cos C-\frac{1}{2}\right)s(s-a)(b-c)}{S_BS_C},$$

because

$$\begin{vmatrix} bS_B & cS_C \\ \cos B+\frac{1}{2} & \cos\mathrm{C}+\frac{1}{2} \end{vmatrix} = \begin{vmatrix} bS_B & cS_C \\ \frac{S_B}{ac}+\frac{1}{2} & \frac{S_C}{ab}+\frac{1}{2} \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} bS_B & cS_C \\ bS_B+\frac{abc}{2} & S_C+\frac{abc}{2} \end{vmatrix}$$
$$=\frac{1}{abc}\begin{vmatrix} bS_B & cS_C \\ \frac{abc}{2} & \frac{abc}{2} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} bS_B & cS_C \\ 1 & 1 \end{vmatrix} = \frac{1}{2}\left(bS_B-cS_C\right)$$
$$=\frac{1}{4}\left(b(a^2-b^2+c^2)-c(a^2+b^2-c^2)\right)$$
$$=\frac{-(b-c)(a+b+c)(b+c-a)}{4}.$$

5. Right triangle at the incenter

Problem 5.1. *Let ABC be a triangle and X the contact point of the $C-$excircle and BC, and Y the intersection of CA and the line parallel to AX through B. Then we have $\angle YIC=90°$.*

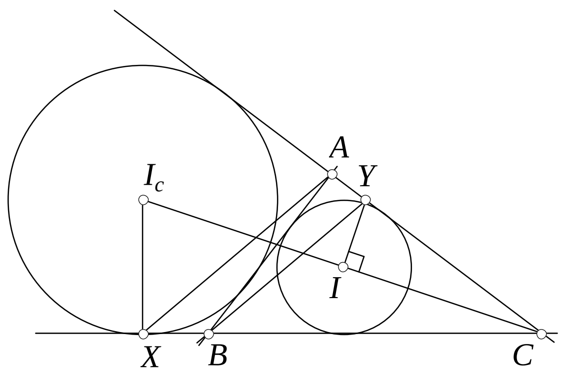

Figure 5

Since $CY:YA=CB:BX=a:s-a$, we have $Y=(a:0:s-a)$. The incenter is $I=(a:b:c)$, with sum of coordinates (weight) $a+b+c=2s$. The infinite point of line IY is $(2a-a:-b:2(s-a)-c)=(a:-b:b-a)$, the same as the infinte point of the line $bx+ay=0$, which is the external angle bisector of angle C. Therefore IY e IC are perpendicular.

6. The same homothety center

Problem 6.1. *Let ABC be a triangle and P,Q two points on line segment BC. Let (P_B), (P_C), (Q_B) and (Q_C) be the incircles of the triangles ABP, APC, ABQ and AQC. Then the external center of homothety of the circles (P_B) and (Q_C) is the same as the external center of homothety of the cricles P_C y Q_B.*

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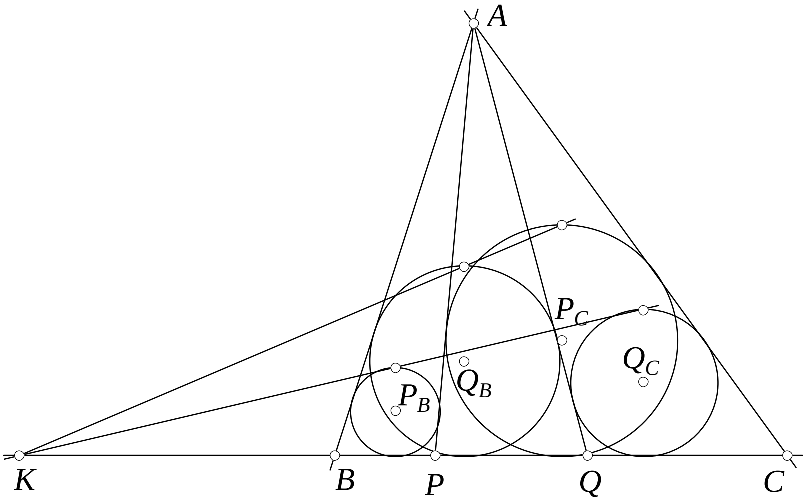
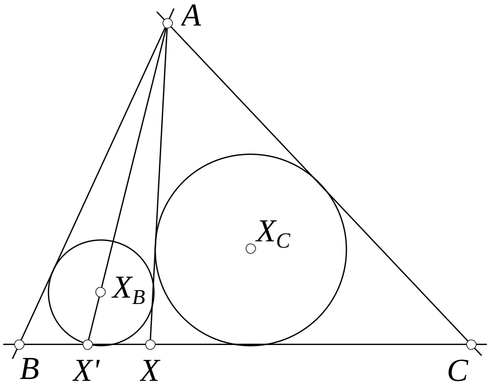


Solution. We first use barycentric coordinates to prove some lemmas:

Lemma 6.1. *If X lies on BC and $BX : XC = t : 1-t$, then the incenter X_B of triangle ABX has homogeneous barycentric coordinates*

$$X_B = (at : AX + (1-t)c : ct).$$



Proof. If $X' = AX_B \cap BC$, we have $X = (0 : 1-t : t)$ and, by the bisector theorem, $BX' : X'X = AB : AX = c : AX$ we get$X' = (0 : AX + (1-t)c : tc)$. On the other side, since X_B lies on the angle bisector $BI : cx - az = 0$, we find the intersection point $X_B = (at : AX + (1-t)c : ct)$.

Symmetrically, the incenter X_C of triangle AXC has homogeneous barycentric coordinates $X_C = ((1-t)a : (1-t)b : bt + AX)$.

Lemma 6.2. *If X lies on BC and $BX : XC = t : 1-t$, then we have $AX^2 = tb^2 + (1-t)c^2 - t(1-t)a^2$.*

Proof. From Stewart theorem for cevian AX,

$$\begin{aligned} &a \cdot \left(AX^2 + BX \cdot XC\right) = b^2 \cdot BX + c^2 \cdot XC \\ \Rightarrow &a \cdot \left(AX^2 + ta \cdot (1-t)a\right) = b^2 \cdot ta + c^2 \cdot (1-t)a \\ \Rightarrow &AX^2 = tb^2 + (1-t)c^2 - t(1-t)a^2. \end{aligned}$$

Lemma 6.3. *If X lies on BC and $BX : XC = t : 1-t$, then*

$$AX^2 - (tb - (1-t)c)^2 = (1-t)t(a+b+c)(b+c-a).$$

Proof. We sum the corresponding sides of the identities

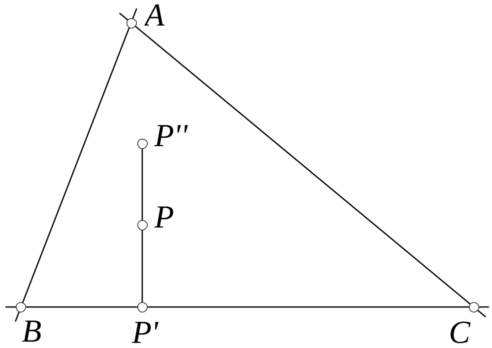
$$(tb-(1-t)c)^2 = t^2b^2+(1-t)^2c^2-2t(1-t)bc$$
$$(1-t)t(a+b+c)(b+c-a) = (1-t)t(b^2+c^2+2bc-a^2)$$

and we get

$$\begin{aligned}&(tb-(1-t)c)^2+(1-t)t(a+b+c)(b+c-a)\\ =&tb^2+(1-t)c^2-t(1-t)a^2\\ =&AP^2.\end{aligned}$$

Lemma 6.4. *Let ABC be a triangle and $P=(u:v:w)$ a point, P' the feet of the perpendicular to BC through P, and P'' the reflection of P' on P. Then P'' has coordinates*

$$P''=(2a^2u : a^2v-uS_C : a^2w-uS_B).$$



Proof. We calculate first the coordinates of P', by finding the line through P and P and $(-a^2:S_C:S_B)$, the infinite point of a perpendicular to BC, then finding its intersection with line $BC: x=0$. To do that, we calculate the determinant

$$\begin{vmatrix} 0 & y & z\\ u & v & w\\ -a^2 & S_C & S_B\end{vmatrix} = 0 \Rightarrow (S_Bu+a^2w)y=(S_Cu+a^2v)z,$$

giving the point $P'=(0:S_Cu+a^2v:S_Bu+a^2w)$. The sum of coordinates of P' is $a^2(u+v+w)$, hence the coordinates of P'' follow from the algebraic relation

$$\begin{aligned}P''=2P-P'=&2(a^2u,a^2v,a^2w)-(0,S_Cu+a^2v,S_Bu+a^2w)\\ =&(2a^2u,a^2v-S_Cu,a^2w-S_Bu).\end{aligned}$$

Now to solve our problem we take $P=(0:1-p:p)$ and $Q=(0:1-q:q)$. Then by Lemma 1, $P_B=(ap:AP+c(1-p):cp)$ and $Q_C=(a(1-q):b(1-q):AQ+bq)$. If P'_B and Q'_C are the orthogonal projections of P and P' on BC and P''_B,Q''_C are their reflection on P_B, Q_C, respectively, obtenemos:

$$P''_B=\left(2pa^2 : aAP+(1-p)ac-pS_C, 2p(s-a)(s-c)\right),$$
$$Q''_C=\left(2(1-q)a^2 : 2(1-q)(s-a)(s-b) : aAQ+qab-(1-q)S_B\right).$$

The line $P''_BQ''_C$ instersects BC at the point

$$K_{PQ}=(0:-(1-q)(AP-pb+(1-p)c):p(AQ+qb-(1-q)c)).$$

Symmetrically, if we consider the points P_C and Q_B we get the that the corresponding line $P''_CQ''_B$ instersects BC at the point

$$K_{QP}=(0:-(1-p)(AQ-qb+(1-q)c):q(AP+pb-(1-p)c)).$$

Since points of the form $(0 : m : n)$ and $(0 : m' : n')$ are the same if and only if $mn' = m'n$, to prove $K_{PQ} = K_{QP}$ we only need that

$$q(1-q)\left(AP^2 - (pb - (1-p)c)^2\right) = p(1-p)\left(AQ^2 - (qb - (1-q)c)^2\right),$$

and this follows inmmediatly from Lemma 4.

References

[1] P. Yiu, *Introduction to the Geometry of the Triangle*, 2001, new version of 2013, `http://math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry130411.pdf`.