International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
June 2016, Volume 1, No.2, pp.40-47
Received 1 March 2016. Published on-line 20 March 2016
web: http://www.journal-1.eu/

Some problems around the Dao's theorem on six circumcenters associated with a cyclic hexagon configuration

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Abstract. In this paper we introduce some problems around the Dao's theorem on six circumcenters associated with a cyclic hexagon. By using the complex coordinates, we give a proof of a problem in this configuration.

Keywords. Dao's theorem, concurrent, circumcenter, cyclic hexagon, concyclic, radical center, circles.

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

In 2013, O. T. Dao published without proof a remarkable theorem with title *Another seven circles theorem* in Cut the Knot [1], a free site for popular expositions of many topics in mathematics [2]. The calculation of barycentric coordinate for concurrence given by N. Dergiades takes more than 72 pages A4 [3]. In 09-2014, N. Dergiades gave an elegant proof of this theorem and renamed this theorem: *Dao's theorem on six circumcenters associated with a cyclic hexagon*. In 10-2014, T. Cohl, a Taiwan student, gave a synthetic proof for this theorem. Two proofs were published in the Forum Geometricorum journal [4] [5]. We introduce some nice problems around the configuration of Dao's theorem in §2. In §3, we use a lemma of Dergiades to prove a problem proposed by O. T. Dao.

In this paper we consider the following configuration: *Let $L_1, L_2, L_3, L_4, L_5, L_6$ be six lines and let $P_{ij} = L_i \cap Lj$, such that $P_{12}, P_{23}, P_{34}, P_{45}, P_{56}, P_{61}$ are concyclic. Let (O_{ijk}) be circle (P_{ij}, P_{jk}, P_{ik}) with center O_{ijk}. Let (O_{ijk}) meets (O_{jkh}) again at P'_{jk}. Taking subscripts modulo 6.*

Theorem 1 (Dao). *$O_{123}O_{456}$, $O_{234}O_{561}$, $O_{345}O_{612}$ are concurrent.*

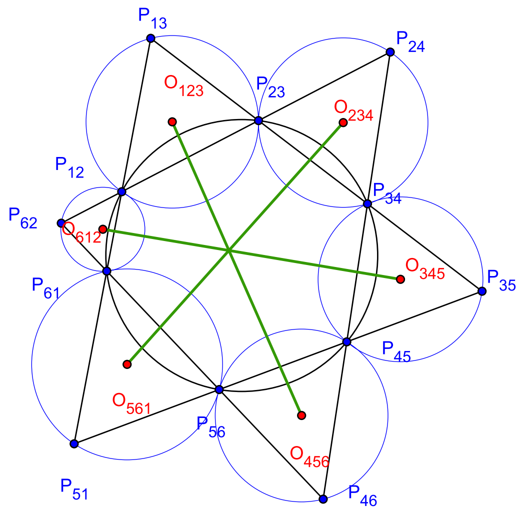

Figure 1. Dao's theorem on six circumcenters associated with a cyclic hexagon configuration

2. Some problems around Dao's theorem on six circumcenter configuration

Problem 1 ([7]). *The radical center of three circles (O_{123}), (O_{345}), (O_{561}) coincides with the radical center of three circles (O_{234}), (O_{456}), (O_{612})*

Problem 2 ([8]). *$\triangle O_{123}O_{345}O_{561}$ and $\triangle P_{24}P_{46}P_{62}$ are orthologic.*

Problem 3 ([8]). *Six circles $(P_{12}P_{23}P_{24})$, $(P_{24}P_{34}P_{35})$, $(P_{35}P_{45}P_{46})$, $(P_{46}P_{56}P_{51})$, $(P_{51}P_{61}P_{62})$, $(P_{62}P_{12}P_{13})$ have a radical center.*

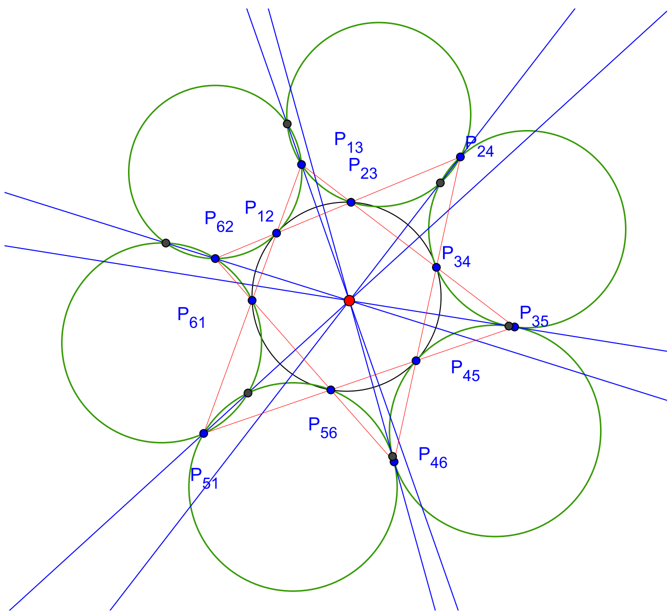

Figure 2. Six circles have a radical center

Problem 4 ([9]). *$P_{12}P_{45}$, $P_{23}P_{56}$, $P_{34}P_{61}$ are concurrent. Then six circumcenters O_{123}, O_{234}, O_{345}, O_{456}, O_{561}, O_{612} lie on a conic.*

Problem 5 ([10]). *$P_{12}P_{45}$, $P_{23}P_{56}$, $P_{34}P_{61}$ are concurrent. Then six circles (O_{123}), (O_{234}), (O_{345}), (O_{456}), (O_{561}), (O_{612}) have a radical center.*

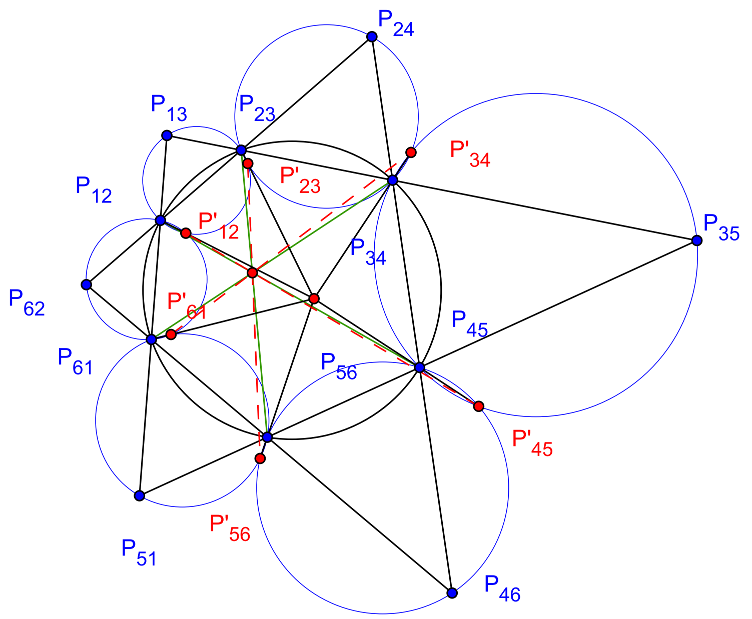

FIGURE 3. Six circles have a radical center

Problem 6 ([11]). *Let $P_{12}P_{45}$, $P_{23}P_{56}$, $P_{34}P_{61}$ are concurrent. Then P'_{12}, P'_{23}, P'_{34}, P'_{45}, P'_{56}, P'_{61} are concyclic.*

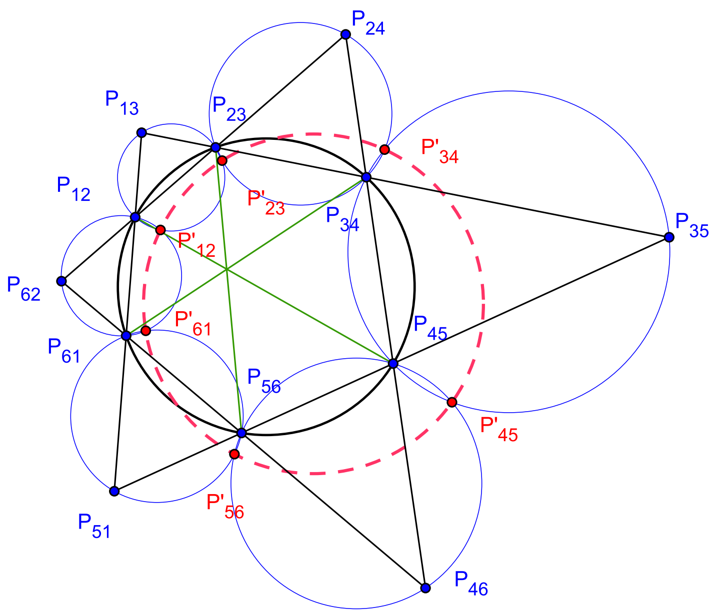

FIGURE 4. P'_{12}, P'_{23}, P'_{34}, P'_{45}, P'_{56}, P'_{61} are concyclic

Problem 7 ([11]). *Let $P_{12}P_{45}$, $P_{23}P_{56}$, $P_{34}P_{61}$ are concurrent. $L'_i = P'_{i,i-1}P'_{i,i+1}$. We define L'_{ij} and (O'_{ijk}), similar to L_{ij} and (O_{ijk}). (O'_{123}), (O'_{234}), (O'_{345}), (O'_{456}), (O'_{561}), (O'_{612}) intersect (O'_{612}), (O'_{123}), (O'_{234}), (O'_{345}), (O'_{456}), (O'_{561}) at P''_{12}, P''_{23}, P''_{34}, P''_{45}, P''_{56}, P''_{61} other than P'_{12}, P'_{23}, P'_{34}, P'_{45}, P'_{56}, P'_{61}.*
Then six circles $(P_{i,i+1}P'_{i,i+1}P''_{i,i+1})$ are coaxial.

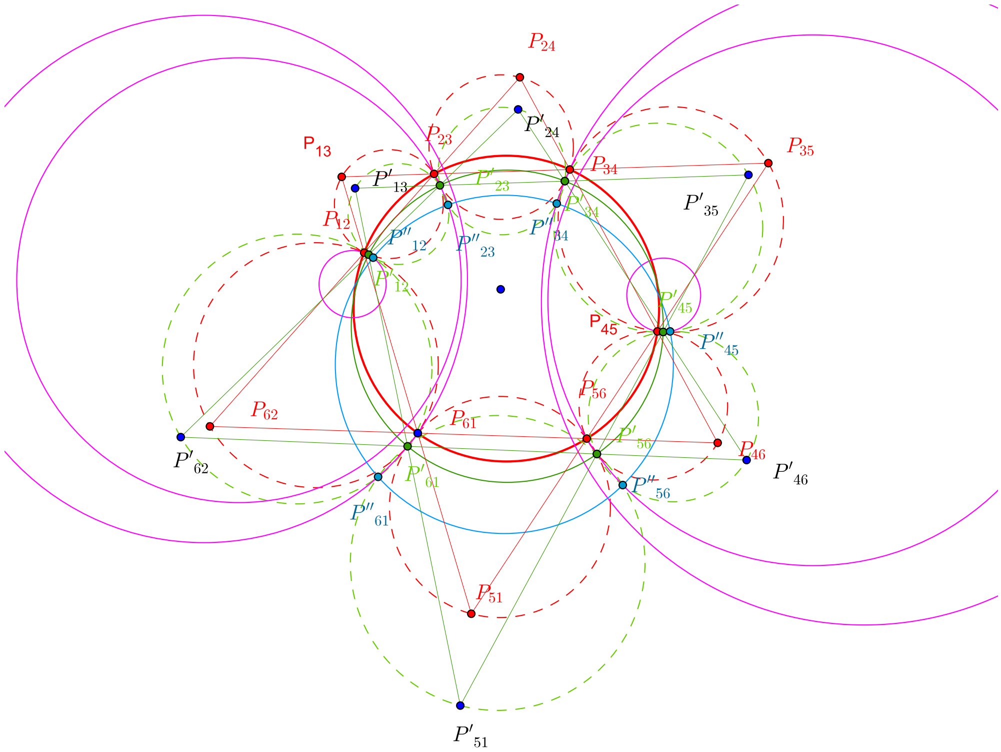

FIGURE 5. Six circles are coaxial

3. Proof of problem 1

We introduce a solution of problem 1 by complex coordinates. In the complex plane, let (O) be an unit circle. Six points on (O) have affixes: $P_{12}(a)$, $P_{23}(y)$, $P_{34}(c)$, $P_{45}(x)$, $P_{56}(b)$, $P_{61}(z)$ the circumcenters have affixes: $O_{612}(a_b)$, $O_{123}(a_c)$, $O_{456}(b_c)$, $O_{561}(b_a)$, $O_{234}(c_a)$, $O_{345}(c_b)$.

We formulate equations of the circumcircle and their radical centers.

Lemma 1 ([4]-Dergiades). *A, B, C, D are the points on an unit circle that have affixes a, b, c, d . $AB \cap CD = E$ then the circumcenter of (ACE) has affixes: $\frac{ac(b-d)}{ab-cd}$*

From lemma 1, we have the affixes of the circumcenters as follows:

$O_{123} = \frac{ay(z-c)}{az-cy}$; $O_{345} = \frac{cx(y-b)}{cy-bx}$; $O_{561} = \frac{bz(x-a)}{bx-az}$

$O_{234} = \frac{cy(x-a)}{cx-ay}$; $O_{456} = \frac{bx(z-c)}{bz-cx}$); $O_{612} = \frac{az(y-b)}{ay-bz}$

Then the equations of the circles $(O_{123}), (O_{345}), (O_{561})$ are:

$$(O_{123}) : |w - \frac{ay(z-c)}{az-cy}| = \frac{|a-y|}{|az-cy|}$$

$$(O_{345}) : |w - \frac{cx(y-b)}{cy-bx}| = \frac{|c-x|}{|cy-bx|}$$

$$(O_{561}) : |w - \frac{bz(x-a)}{bx-az}| = \frac{|b-z|}{|bx-az|}$$

Lemma 2. *In complex plane, let $(O_1), (O_2), (O_3)$ be three circles that have equations: $|w-a_1| = R_1$, $|w-a_2| = R_2$, $|w-a_3| = R_3$. Then the radical center of them has affix:*

$$\frac{(a_1\overline{a_1} - R_1^2)(a_2-a_3) + (a_2\overline{a_2} - R_2^2)(a_3-a_1) + (a_3\overline{a_3} - R_3^2)(a_1-a_2)}{\overline{a_1}(a_2-a_3) + \overline{a_2}(a_3-a_1) + \overline{a_3}(a_1-a_2)}$$

Back to the main proof:

Let $T(t)$ be radical center of $(O_{123}), (O_{561}), (O_{456})$. From lemma 2 we have:

$$t = \frac{(b_a\overline{b_a} - R_{b_a}^2)(c_b-a_c) + (c_b\overline{c_b} - R_{c_b}^2)(a_c-b_a) + (a_c\overline{a_c} - R_{a_c}^2)(b_a-c_b)}{\overline{b_a}(c_b-a_c) + \overline{c_b}(a_c-b_a) + \overline{a_c}(b_a-c_b)}$$

For ease, we divided the solutions into 2 steps.

Step 1: Formulate numerator of t

Note that $|a| = 1$ so $\overline{a} = \frac{1}{a}$

$$b_a\overline{b_a} - R_{b_a}^2 = \frac{bz(x-a)}{bx-az} \cdot \frac{\frac{1}{bz}\cdot(\frac{1}{x} - \frac{1}{a})}{\frac{1}{bx} - \frac{1}{az}} - \frac{\frac{1}{b} - \frac{1}{z}}{\frac{1}{bx} - \frac{1}{az}} \cdot \frac{b-z}{bx-az}$$

$$= \frac{bz(x-a)}{bx-az} \cdot \frac{x-a}{bx-az} - \frac{b-z}{bx-az} \cdot \frac{ax(b-z)}{bx-az} = \frac{bz(x-a)^2 - ax(z-b)^2}{(bx-az)^2}$$

$$= \frac{zx(bx-az) - ab(bx-az)}{(bx-az)^2} = \frac{zx-ab}{bx-az}$$

Therefore,

$$(c_b - a_c)(b_a\overline{b_a} - R_{b_a}^2) = \frac{zx-ab}{bx-az} \cdot \left(\frac{cx(y-b)}{cy-bx} - \frac{ay(z-c)}{az-cy}\right)$$

$$= \frac{zx-ab}{(bx-az)(az-cy)(cy-bx)} \cdot [cx(y-b)(az-cy) - ay(z-c)(cy-bx)]$$

$$= \frac{zx-ab}{(bx-az)(az-cy)(cy-bx)} \cdot [cx(ayz - abz - cy^2 + bcy) - ay(cyz - bzx - c^2y + bcx)]$$

$$= \frac{zx-ab}{(bx-az)(az-cy)(cy-bx)} \cdot (acxyz - abcxz - c^2xy^2 + bc^2xy - acy^2z + abxyz + ac^2y^2 - abcxy)$$

$$= \frac{acx^2yz^2 - abcx^2z^2 - c^2x^2y^2z + bc^2x^2yz - acxy^2z^2 + abx^2yz^2 + ac^2xy^2z - abcx^2yz - a^2bcxyz}{(bx-az)(az-cy)(cy-bx)}$$

$$+ \frac{a^2b^2cxz + abc^2xy^2 - ab^2c^2xy + a^2bcy^2z - a^2b^2xyz - a^2bc^2y^2 + a^2b^2cxy}{(bx-az)(az-cy)(cy-bx)} ;$$

$(a_c - b_a)(c_b\overline{c_b} - R^2_{c_b}) = \dfrac{(xy-bc)}{bx-az)(az-cy)(cy-bx)}.[ay(z-c)(bx-az)-bz(x-a)(az-cy)]$

$=\dfrac{xy-bc}{(bx-az)(az-cy)(cy-bx)}.[ay(bxz - az^2 - bcx + acz) - bz(a - cxy - a^2z + acy)]$

$=\dfrac{xy-bc}{bx-az)(az-cy)(cy-bx)}.(abxyz-a^2yz^2-abcxy+a^2cyz-abxz^2+bcxyz+a^2bz^2-abcyz)$

$=\dfrac{abx^2y^2z-a^2xy^2z^2-abcx^2y^2+a^2cxy^2z-abx^2yz^2+bcx^2y^2z+a^2bxyz^2-abcxy^2z-ab^2cxyz}{(bx-az)(az-cy)(cy-bx)}$

$+\dfrac{a^2bcyz^2+ab^2c^2xy-a^2bc^2yz+ab^cxz^2-b^2c^2xyz-a^2b^2cz^2+ab^2c^2yz}{(bx-az)(az-cy)(cy-bx)}$;

$(b_a - c_b)(a_c\overline{a_c} - R^2_{a_c}) = \dfrac{yz-bc}{(bx-az)(az-cy)(cy-bx)}.[bz(x-a)(cy-bx)-cx(y-b)(bx-az)]$

$=\dfrac{yz-bc}{(bx-az)(az-cy)(cy-bx)}.[bz(cxy - bx^2 - acy + abx) - cx(bxy - ayz - b^2x + abz)]$

$=\dfrac{yz-bc}{(bx-az)(az-cy)(cy-bx)}.(bcxyz-b^2x^2z-abcyz+ab^2xz-bcx^2y+acxyz+b^2cx^2-abcxz)$

$=\dfrac{bcxy^2z^2-b^2x^2yz^2-abcy^2z^2+ab^2xyz-bcx^2y^2z+acxy^2z^2+b^2cx^2yz-abcxyz^2-abc^2xyz}{(bx-az)(az-cy)(cy-bx)}$

$+\dfrac{ab^2cx^2z+a^2bc^2yz-a^2b^2cxz+abc^2x^2y-a^2c^2xyz-ab^2c^2x^2+a^2bc^2xz}{(bx-az)(az-cy)(cy-bx)}$;

So numerator of $t =$

$\dfrac{xy^2z^2(bc-a^2)+x^2yz^2(ca-b^2)+x^2y^2z(ab-c^2)+x^2yzbc(b+c-a)+xy^2zca(c+a-b)+xyz^2ab(a+b-c)}{(bx-az)(az-cy)(cy-bx)}$
$+\dfrac{-xyz[a^2b^2+b^2c^2+c^2a^2+abc(a+b+c)]-abc(x^2y^2+y^2z^2+z^2x^2)}{(bx-az)(az-cy)(cy-bx)}$
$+\dfrac{abc^2xy(x+y)+a^2bcyz(y+z)+ab^2czx(z+x)+ab^2c^2yz+a^2bc^2zx+a^2b^2cxy-ab^2c^2x^2-a^2bc^2y^2-a^2b^2cz^2}{(bx-az)(az-cy)(cy-bx)}$

Step 2: Formulate denominator of $t = \overline{b_a}(c_b - a_c) + \overline{c_b}(a_c - b_a) + \overline{a_c}(b_a - c_b)$

$\overline{b_a}(c_b - a_c) = \dfrac{\frac{1}{bz}(\frac{1}{x}-\frac{1}{a})}{\frac{1}{bx}-\frac{1}{az}}.[\dfrac{cy(y-b)}{cy-bx} - \dfrac{ay(z-c)}{az-cy}]$
$=\dfrac{x-a}{(bx-az)(az-cy)(cy-bx)}.(cay - abcxz + bc^2xy - c^2xy^2 - acy^2z - abcxy + ac^2y^2 + abxyz)$
$=\dfrac{acx^2yz-abcx^2z+bc^2x^2y-c^2x^2y^2-acxy^2z-abcx^2y+ac^2xy^2+abx^2yz-a^2cxyz}{(bx-az)(az-cy)(cy-bx)}$
$+\dfrac{a^2bcxz-abc^2xy+ac^2xy^2+a^2cy^2z+a^2bcxy-a^2c^2y^2-a^2bxyz}{(bx-az)(az-cy)(cy-bx)}$;

$\overline{c_b}(a_c - b_a) = \dfrac{(y-b)(abxyz+a^2cyz-abcxy-a^2yz^2-abxz^2+bcxyz+a^2bz^2-abcyz)}{(bx-az)(az-cy)(cy-bx)}$

$= \dfrac{abxy^2z+a^2cy^2z-abcxy^2-a^2y^2z^2-abxyz^2+bcxy^2z+a^2byz^2-abcy^2z-ab^2xyz-a^2bcyz}{(bx-az)(az-cy)(cy-bx)}$

$+ \dfrac{ab^2cxy+a^2byz^2+ab^2xz^2-b^2cxyz-a^2b^2z^2+ab^2cyz}{(bx-az)(az-cy)(cy-bx)}$;

$\overline{a_c}(b_a - c_b) = \dfrac{(z-c)(bcxyz-abcyz-b^2x^2z+ab^2xz-bcx^2y+b^2cx^2-abcxz+acxyz)}{(bx-az)(az-cy)(cy-bx)}$

$= \dfrac{bcxyz^2-abcyz^2-b^2z^2x^2+ab^2xz^2-bcx^2yz+b^2cx^2z-abcxz^2+acxyz^2-bc^2xyz}{(bx-az)(az-cy)(cy-bx)}$

$+ \dfrac{abc^2yz+b^2cx^2z-ab^2cxz+bc^2x^2y-b^2c^2x^2+abc^2xz-ac^2xyz}{(bx-az)(az-cy)(cy-bx)}$

So denominator of $t =$

$\dfrac{x^2yz(ab+ac-bc)+xy^2z(bc+ba-ca)+xyz^2(ca+cb-ab)-xyz[ab(a+b)+bc(b+c)+ca(c+a)]}{(bx-az)(az-cy)(cy-bx)}$

$+ \dfrac{2bcx^2(bz+cy)+2cay^2(cx+az)+2abz^2(ay+bx)-abc[xy(x+y)+yz(y+z)+zx(z+x)]}{(bx-az)(az-cy)(cy-bx)}$

$+ \dfrac{abcxy(a+b-c)+abcyz(b+c-a)+abczx(c+a-b)-c^2x^2y^2-b^2z^2x^2-a^2y^2z^2-x^2b^2c^2-y^2c^2a^2-z^2a^2b^2}{(bx-az)(az-cy)(cy-bx)}$

Let $f(a, y, c, x, b, z) =$ affix of T. If we replace $(a, y, c, x, b, z) \mapsto (y, c, x, b, z, a)$ then (O_{123}), (O_{345}), (O_{561}) become (O_{234}), (O_{456}), (O_{612}) then radical center of (O_{123}), (O_{345}), (O_{561}) becomes radical center of (O_{234}), (O_{456}), (O_{612}). Hence radical center of (O_{234}), (O_{456}), (O_{612}) has affix $f(y, c, x, b, z, a)$. It is easy to verify that $f(a, y, c, x, b, z) = f(y, c, x, b, z, a)$. Therefore, radical center of (O_{123}), (O_{345}), (O_{561}) coincides with radical center of (O_{234}), (O_{456}), (O_{612}).

REFERENCES

[1] Alexander Bogomolny. "Cut-the-knot.org: Another seven circles theorem" `http://www.cut-the-knot.org/m/Geometry/AnotherSevenCircles.shtml`

[2] `https://en.wikipedia.org/wiki/Cut-the-Knot`

[3] Nikolaos Dergiades, Advanced Plane Geometry, message 1539, August 29, 2014 `https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/1539`

[4] Nikolaos Dergiades, *Dao's Theorem on Six Circumcenters associated with a Cyclic Hexagon*, Forum Geom, 14(2014). p. 243–246. `http://forumgeom.fau.edu/FG2014volume14/FG201424.pdf`

[5] Telv Cohl, *A purely synthetic proof of Dao's theorem on six circumcenters associated with a cyclic hexagon*, Forum Geom, 14(2014), p. 261–264. `http://forumgeom.fau.edu/FG2014volume14/FG201429.pdf`

[6] John Rigby, Brief notes on some forgotten geometrical theorems, Mathematics and Informatics Quarterly (1997), p. 156-158

[7] O. T. Dao, Advanced Plane Geometry, message 2391, March 7, 2015 `https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/1714`

[8] O. T. Dao, Advanced Plane Geometry, message 2410, March 7, 2015 `https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/2410`

[9] O. T. Dao, Advanced Plane Geometry, message 2633, July 7, 2015 `https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/2633`

[10] Seiichi Kirikami, Advanced Plane Geometry, message 1713, September 14, 2014
`https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/2391`

[11] D. Q. Ngo, Advanced Plane Geometry, message 2628, July 16, 2015
`https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/2628`