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Computer Discovered Mathematics: Notable Circles

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Abstract. By using the computer program "Discoverer" we study a few notable circles of triangle.

Keywords. triangle geometry, notable point, Euclidean geometry, computer discovered mathematics, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

Let ABC be a triangle with side lengths $a = BC, b = CA$ and $c = AC$. Denote by $C(X, Y)$ the circle having the segment joining points X and Y as its diameter.

We denote by $X(n)$ the Kimberling points [8] of triangle ABC. Hence, $X(2)$ is the Centroid, $X(3)$ is the Circumcenter, $X(4)$ is the Orthocenter and $X(5)$ is the Nine-Point Center of triangle ABC.

We will study the circles $\mathcal{C} = \{C(X(i), X(j)), 2 \le i < j \le 5\}$. Denote $C(X(i), X(j)$ as $C(i, j)$ for short. Note that $C(2, 4)$ is the well known orthocentroidal circle (See [13], Orthocentroidal circle). It seems that the other circles of $\mathcal{C}$ are not investigated.

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We use barycentric coordinates [1] - [16]. Denote by Δ the area of triangle ABC and denote

$$E = \frac{a^6 + b^6 + c^6 + 3a^2b^2c^2 - a^4b^2 - a^4c^2 - a^2c^4 - a^2b^4 - b^2c^4 - b^4c^2}{48\Delta}.$$

Then E is the radius of circle $C(2,5)$) and the other radii of circles in $\mathcal{C}$ follow. (See [13], Euler Line).

We use the computer program "Discoverer" created by the authors.

2. Circle $C(2,3)$

It is known ([8]) that the midpoint of Centroid and Circumcenter is point X(549). The distance between Centroid an Circumcenter is $4E$ ([13], Euler line). Hence, the center of circle $C(2,3)$ is point X(549) and the radius is $2E$.

The "Discoverer" easily discovers new theorems about Circle $C(2,3)$.

Theorem 2.1. *The Circle $C(2,3)$ is the*

(1) *Orthocentroidal Circle of the Medial Triangle.*
(2) *Orthocentroidal Circle of the Half-Anticevian Triangle of the Centroid.*
(3) *Circumcircle of the Half-Circumcevian Triangle of the Centroid.*
(4) *Circumcircle of the Fourth Brocard Triangle of the Medial Triangle.*
(5) *Orthocentroidal Circle of the Half-Median Triangle of the Antimedial Triangle.*
(6) *Complement of the Orthocentroidal Circle.*
(7) *Image of the Orthocentroidal Circle under Homothety with Center the Centroid and Ratio $-\frac{1}{2}$.*
(8) *Inverse Circle of the Orthocentroidal Circle of the Half-Median Triangle in the Orthocentroidal Circle.*

Theorem 2.2. *The Circle $C(2,3)$ is congruent with the*

(1) *Orthocentroidal Circle of the Euler Triangle.*
(2) *Nine-Point Circle of the Fourth Brocard Triangle.*
(3) *Nine-Point Circle of the Fourth Brocard Triangle of the Johnson Triangle.*
(4) *Excentral Circle of the Half-Median Triangle of the Fourth Brocard Triangle.*
(5) *Excentral Circle of the Fourth Brocard Triangle of the Half-Median Triangle.*
(6) *Antimedial Circle of the Fourth Brocard Triangle of the Half-Median Triangle.*
(7) *Orthocentroidal Circle of the Euler Triangle of an arbitrary triangle center P.*
(8) *Image of the Orthocentroidal Circle under the Homothety with Center an arbitrary triangle center P and Ratio $-\frac{1}{2}$.*

Figure 1 illustrates part (1) of Theorem 2.1. In figure 1,

- c_1 is the Circle $C(2,3)$,
- $EaEbEc$ is the Euler triangle,
- c_2 is the Orthocentroidal Circle of the Euler Triangle.

Circles c_1 and c_2 are congruent.

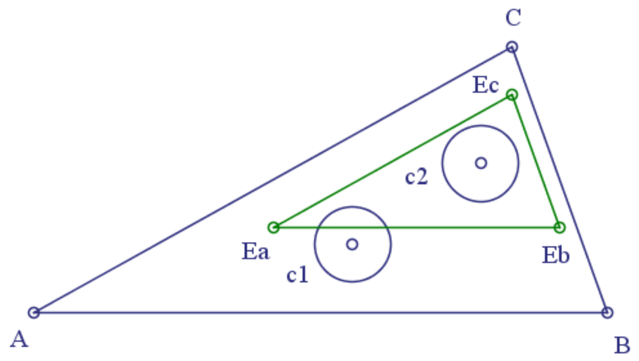

Figure 1.

Theorem 2.3. *The following circles are orthogonal to Circle (2,3):*

(1) *Stevanovic Circle of the Medial Triangle.*
(2) *Lester Circle of the Medial Triangle.*
(3) *Parry Circle of the Fourth Brocard Triangle.*
(4) *Lester Circle of the Euler Triangle of the Nine-Point Center.*
(5) *Lester Circle of the Euler Triangle of the Steiner Point.*
(6) *Stevanovic Circle of the Half-Circumcevian Triangle of the Centroid.*
(7) *Parry Circle of the Half-Circumcevian Triangle of the Centroid.*
(8) *Parry Circle of the Half-Circumcevian Triangle of the Orthocenter.*
(9) *Orthocentroidal Circle of the Triangle of the Circumcenters of the Triangulation Triangles of the Center of the Orthocentroidal Circle.*
(10) *Parry Circle of the Fourth Brocard Triangle of the Antimedial Triangle.*
(11) *Brocard Circle of the First Brocard Triangle of the Honsberger Triangle.*
(12) *Parry Circle of the Fourth Brocard Triangle of the Half-Median Triangle.*
(13) *Circle passing through the Centroid, Outer Fermat Point and Parry Point.*
(14) *Circle passing through the Centroid, Retrocenter and Steiner Point.*
(15) *Circle passing through the Centroid, Kiepert Center and Symmedian Point.*
(16) *Circle passing through the Circumcenter, Gibert Point and Kosnita Point.*
(17) *Circle passing through the Center of the Brocard Circle, Circumcenter and Tarry Point.*
(18) *Image of the Orthocentroidal Circle under the Homothety with Center the Circumcenter and Ratio 2/3.*
(19) *Image of the Orthocentroidal Circle under the Homothety with Center the Orthocenter and Ratio 6/5.*
(20) *Inverse Circle of the Radical Circle of the Neuberg Circles in the Second Brocard Circle.*
(21) *Inverse Circle of the Stevanovic Circle of the Half-Median Triangle in the Orthocentroidal Circle.*
(22) *Inverse Circle of the Lester Circle of the Half-Median Triangle in the Orthocentroidal Circle.*

Figure 2 illustrates parts (2) and (4) of Theorem 2.3. In figure 2,

- c_1 is the Circle $C(2,3)$,
- c_2 is the Lester Circle of the Medial Triangle,
- c_3 is the Lester Circle of the Euler Triangle of the Nine-Point Center.

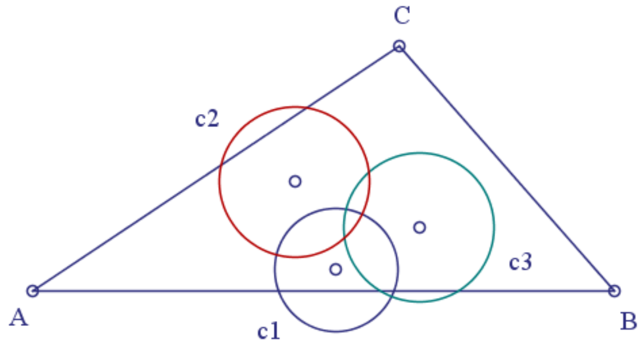

FIGURE 2.

Theorem 2.4. *The following points lie on the Circle $C(2,3)$:*

(1) *Parry Point of the First Brocard Triangle.*
(2) *Euler Reflection Point of the Euler Triangle of the Tarry Point.*
(3) *Parry Point of the Euler Triangle of the Steiner Point.*
(4) *Tarry Point of the Half-Circumcevian Triangle of the Centroid.*
(5) *Steiner Point of the Half-Circumcevian Triangle of the Centroid.*
(6) *Euler Reflection Point of the Half-Circumcevian Triangle of the Spieker Center.*
(7) *Euler Reflection Point of the Half-Circumcevian Triangle of the Outer Napoleon Point.*
(8) *Euler Reflection Point of the Half-Circumcevian Triangle of the Inner Napoleon Point.*
(9) *Euler Reflection Point of the Half-Circumcevian Triangle of the Third Brocard Point.*
(10) *Euler Reflection Point of the Half-Circumcevian Triangle of the Outer Vecten Point.*
(11) *Euler Reflection Point of the Half-Circumcevian Triangle of the Inner Vecten Point.*
(12) *Centroid of the Half-Circumcevian Triangle of the First Brocard Point.*
(13) *Centroid of the Half-Circumcevian Triangle of the Second Brocard Point.*
(14) *Centroid of the Half-Circumcevian Triangle of the First Beltrami Point.*
(15) *Gibert Point of the Half-Circumcevian Triangle of the Centroid.*
(16) *Inverse of the Center of the Radical Circle of the Neuberg Circles in the Second Brocard Circle.*

Figure 3 illustrates part (1) of Theorem 2.4. In figure 3,

- c is the Circle $C(2,3)$,
- $B_1B_2B_3$ is the First Brocard triangle,
- P is the Parry point.

We have investigated 193 inverse images of notable points of triangle ABC with respect to circle $C(2,3)$. Of these 14 points are Kimberling centers and the rest of 179 points are new points, not available in Kimberling [8]. See the Supplementary material.

Table 1 gives inverse images wrt circle $C(2,3)$ which are Kimberling centers.

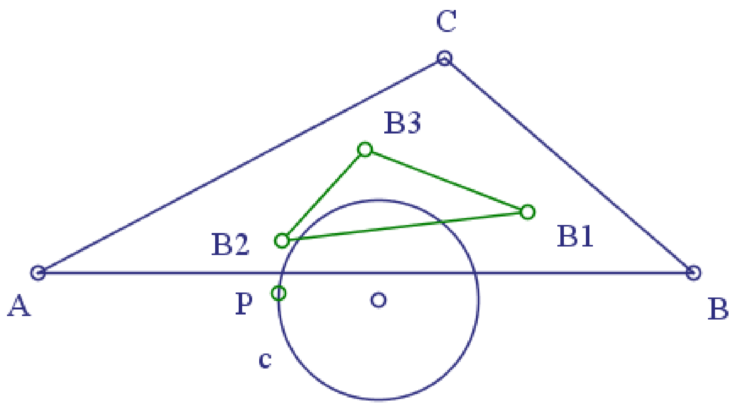

Figure 3.

	Point P	Inverse of Point P wrt circle C(2,3)
1	X(4) Orthocenter	X(631)
2	X(5) Nine-Point Center	X(140)
3	X(20) de Longchamps Point	X(3523)
4	X(21) Schiffler Point	X(404)
5	X(22) Exeter Point	X(7485)
6	X(23) Far-Out Point	X(7496)
7	X(24) Perspector of the Kosnita Triangle and the Orthic Triangle	X(7509)
8	X(25) Product of the Orthocenter and the Symmedian Point	X(7484)
9	X(26) Circumcenter of the Tangential Triangle	X(7516)
10	X(27) Quotient of the Orthocenter and the Spieker Center	X(7573)
11	X(28) Quotient of the Clawson Point and the Spieker Center	X(7523)
12	X(29) Ceva Product of the Incenter and the Orthocenter	X(7572)
13	X(381) Center of the Orthocentroidal Circle	X(5054)
14	X(384) Conway Point	X(7824)

Table 1.

Also, we could investigate properties of the new points listed in the Supplementary material.

The “Discoverer” easily finds relations between the new points and other triangle objects. For example consider the Inverse of the Incenter wrt the Circle $C(2,3)$.

Theorem 2.5. *The first barycentric coordinate of the Inverse of the Incenter wrt the Circle C(2,3) is as follows:*

$4a^2c^4b - b^6c - b^5c^2 + 2b^4c^3 + 2b^3c^4 - b^2c^5 - bc^6 + 24a^5bc + a^4b^2c + a^4bc^2 - 30a^3bc^3 - 30a^3b^3c + 13a^3b^2c^2 + 7a^2b^2c^3 + 4a^2b^4c + 7a^2b^3c^2 + 6ab^5c - 12ab^3c^3 + 6abc^5 - 4a^6b - 4a^6c - 5a^5c^2 - 5a^5b^2 + 5a^4c^3 + 5a^4b^3 + 4a^3c^4 + 4a^3b^4 - a^2c^5 - a^2b^5 + a^7$.

Proof. We use the inversion formula (20) in [5]. □

Theorem 2.6. *Inverse of the Incenter wrt the Circle C(2,3) is the Complement of the Inverse of the Nagel Point in the Orthocentroidal Circle.*

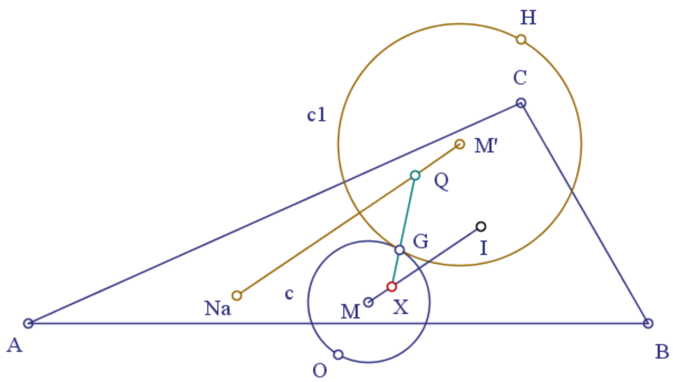

FIGURE 4.

Figure 4 illustrates Theorem 2.6. In figure 4,

- G is the Centroid,
- O is the Circumcenter,
- c is the circle $C(2,3)$,
- M is the center of circle c,
- H is the Orthocenter,
- c_1 is the Orthocentroidal circle $C(2,4)$,
- M' is the center of circle c_1,
- Na is the Nagel point,
- Q is the inverse image of point Na wrt circle c_1,
- I is the Incenter,
- X is the inverse image of point I wrt circle $C(2,3)$.

Then point X is the complement of point Q (that is, the homothetic image of point Q) under the homothety with center G and scale factor $-\frac{1}{2}$. Also, we see that point X lies on the lines IM and QG.

Theorem 2.7. *Denote by $QaQbQc$ the Antipedal triangle of the Inverse of the Nagel Point wrt the Orthocentroidal Circle C(2,4), and denote by Ha, Hb and Hc the orthocenters of triangles $QaBC, QbCA$ and $QcAB$ respectively. Then triangles ABC and $HaHbHc$ are homothetic. The center of the homothety is the Inverse of the Incenter wrt circle $C(2,3)$.*

Figure 5 illustrates Theorem 2.7. In figure 5, Q is the Inverse of the Nagel Point wrt the Orthocentroidal Circle C(2,4), and X is the Inverse of the Incenter wrt circle $C(2,3)$. Point X is the center of homothety of triangles ABC and $HaHbHc$.

Problem 2.1. *Find the scale factor of the homothety given in Theorem 2.7.*

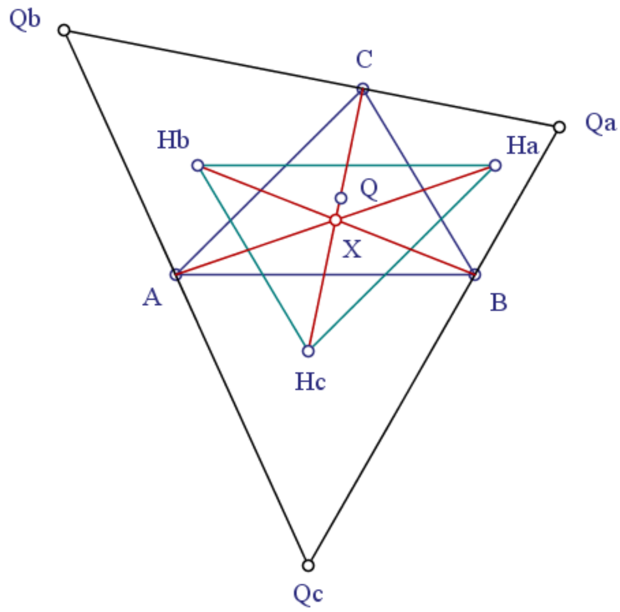

Figure 5.

3. Circle $C(2,4)$

This is the known Orthocentroidal circle [13]. The center of circle $C(2,4)$ is point X(381) and the radius is $4E$.

The "Discoverer" easily discovers theorem about Circle $C(2,4)$.

Theorem 3.1. *The Circle $C(2,4)$ is the*

(1) *Circumcircle of the Fourth Brocard Triangle.*
(2) *Orthocentroidal Circle of the Outer Apollonius Triangle of the Lucas Circles.*
(3) *Orthocentroidal Circle of the Inner Monge Triangle of the Excircles.*
(4) *Orthocentroidal Circle of the Triangle of the Orthocenters of the Triangulation Triangles of the Orthocenter.*
(5) *Orthocentroidal Circle of the Tangential Triangle of the Intouch Triangle.*
(6) *Antimedial Circle of the Medial Triangle of the Fourth Brocard Triangle.*
(7) *Tangential Circle of the Intouch Triangle of the Fourth Brocard Triangle.*
(8) *Nine-Point Circle of the Excentral Triangle of the Fourth Brocard Triangle.*
(9) *Nine-Point Circle of the Antimedial Triangle of the Fourth Brocard Triangle.*
(10) *Nine-Point Circle of the Hexyl Triangle of the Fourth Brocard Triangle.*
(11) *Circle having center at the Center of the Orthocentroidal Circle and passing through the Centroid.*
(12) *Circle having center at the Center of the Orthocentroidal Circle and passing through the Orthocenter.*
(13) *Inverse Circle of the Orthocentroidal Circle in the Stevanovic Circle.*
(14) *Inverse Circle of the Orthocentroidal Circle in the Lester Circle.*

Theorem 3.2. *The Circle $C(2,4)$ is congruent with the*

(1) *Orthocentroidal Circle of the Johnson Triangle.*
(2) *Orthocentroidal Circle of the Circumcevian Triangle of the Circumcenter.*

(3) *Orthocentroidal Circle of the Circumcevian Triangle of the First Brocard Point.*
(4) *Orthocentroidal Circle of the Circumcevian Triangle of the Second Brocard Point.*
(5) *Orthocentroidal Circle of the Circumcevian Triangle of the First Beltrami Point.*
(6) *Orthocentroidal Circle of the Anticevian Euler Triangle of the Centroid.*
(7) *Excentral Circle of the Half-Circumcevian Triangle of the Centroid.*
(8) *Antimedial Circle of the Half-Circumcevian Triangle of the Centroid.*

The Stevanovich circle and the Lester circle are orthogonal with the Circle $C(2.4)$ ([13], Orthocentroidal circle). The "Discoverer has discovered about 500 additional circles orthogonal with the Circle $C(2,4)$. A few of them a listed below.

Theorem 3.3. *The Circle $C(2,4)$ is orthogonal with the*

(1) *Stevanovic Circle.*
(2) *Lester Circle.*
(3) *Stevanovic Circle of the Fourth Brocard Triangle.*
(4) *Parry Circle of the Fourth Brocard Triangle.*
(5) *Parry Circle of the Triangle of Reflections of the Outer Fermat Point in the Sidelines of Triangle ABC.*
(6) *Parry Circle of the Triangle of Reflections of the Inner Fermat Point in the Sidelines of Triangle ABC.*
(7) *Orthocentroidal Circle of the Triangle of Reflections of the First Beltrami Point in the Sidelines of Triangle ABC.*
(8) *Orthocentroidal Circle of the Triangle of Reflections of the Second Beltrami Point in the Sidelines of Triangle ABC.*
(9) *Parry Circle of the Half-Circumcevian Triangle of the Centroid.*
(10) *Circumcircle of the Half-Circumcevian Triangle of the Nine-Point Center.*
(11) *Lester Circle of the Triangle of the Orthocenters of the Triangulation Triangles of the Tarry Point.*
(12) *Lester Circle of the Triangle of the Nine-Point Centers of the Anticevian Corner Triangles of the Centroid.*
(13) *Parry Circle of the Fourth Brocard Triangle of the Antimedial Triangle.*
(14) *Stevanovic Circle of the Intouch Triangle of the Fourth Brocard Triangle.*
(15) *Stevanovic Circle of the Excentral Triangle of the Fourth Brocard Triangle.*
(16) *Stevanovic Circle of the Antimedial Triangle of the Fourth Brocard Triangle.*
(17) *Stevanovic Circle of the Hexyl Triangle of the Fourth Brocard Triangle.*
(18) *Parry Circle of the Second Brocard Triangle of the Fourth Brocard Triangle.*
(19) *Parry Circle of the Fourth Brocard Triangle of the Half-Median Triangle.*
(20) *Circle having center at the Kiepert Center and passing through the Outer Fermat Point.*
(21) *Circle having as its diameter the line segment connecting the Inner Napoleon Point and Outer Napoleon Point.*
(22) *Circle having as its diameter the line segment connecting the Kiepert Center and Symmedian Point.*
(23) *Circle having as its diameter the line segment connecting the Inner Vecten Point and Outer Vecten Point.*

(24) *Circle having as its diameter the line segment connecting the Center of the Lester Circle and Symmedian Point.*
(25) *Circle having as its diameter the line segment connecting the Center of the Lester Circle and Outer Napoleon Point.*
(26) *Circle having as its diameter the line segment connecting the Center of the Lester Circle and Inner Napoleon Point.*
(27) *Circle passing through the Incenter, Inner Fermat Point and Outer Fermat Point.*
(28) *Circle passing through the Incenter, Kiepert Center and Symmedian Point.*
(29) *Circle passing through the Centroid, Outer Fermat Point and Parry Point.*
(30) *Circle passing through the Centroid, Retrocenter and Steiner Point.*
(31) *Circle passing through the Centroid, Kiepert Center and Symmedian Point.*
(32) *Inverse Circle of the Lester Circle in the Stevanovic Circle.*
(33) *Inverse Circle of the Stevanovic Circle in the Lester Circle.*
(34) *Inverse Circle of the Stevanovic Circle of the Fourth Brocard Triangle in the Stevanovic Circle.*
(35) *Inverse Circle of the Parry Circle of the Fourth Brocard Triangle in the Stevanovic Circle.*
(36) *Inverse Circle of the Stevanovic Circle of the Fourth Brocard Triangle in the Lester Circle.*
(37) *Inverse Circle of the Parry Circle of the Fourth Brocard Triangle in the Lester Circle.*

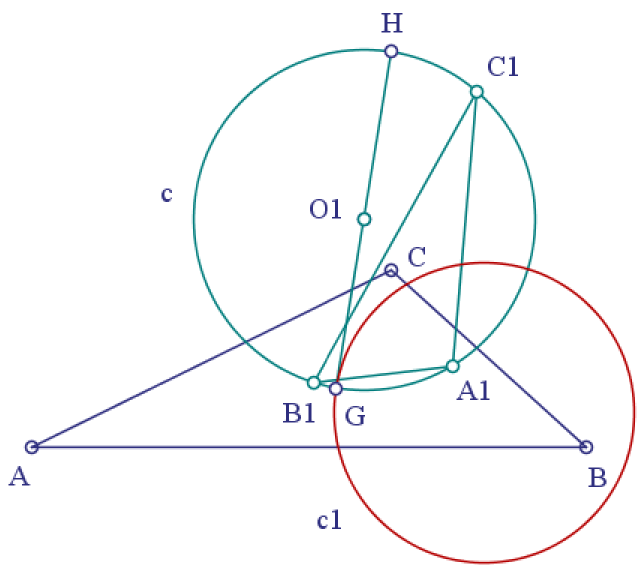

Figure 6.

Figure 6 illustrates part 4 of Theorem 3.3. In figure 6,

- G is the Centroid,
- H is the Orthocenter,
- c is the Orthocentroidal circle $C(2, 4)$,
- O_1 is the center of circle c,
- $A_1B_1C_1$ is the Fourth Brocard Triangle,
- c_1 is the Parry circle of triangle $A_1B_1C_1$.

Theorem 3.4. *The following notable points lie on Circle $C(2.4)$:*

(1) *Tarry Point of the Fourth Brocard Triangle.*
(2) *Steiner Point of the Fourth Brocard Triangle.*
(3) *Euler Reflection Point of the Fourth Brocard Triangle.*
(4) *Gibert Point of the Fourth Brocard Triangle.*
(5) *Parry Point of the Triangle of the Orthocenters of the Triangulation Triangles of the Tarry Point.*
(6) *Euler Reflection Point of the Triangle of the Orthocenters of the Triangulation Triangles of the Steiner Point.*
(7) *Incenter of the Orthic Triangle of the Fourth Brocard Triangle.*
(8) *Feuerbach Point of the Excentral Triangle of the Fourth Brocard Triangle.*
(9) *Feuerbach Point of the Antimedial Triangle of the Fourth Brocard Triangle.*
(10) *Feuerbach Point of the Hexyl Triangle of the Fourth Brocard Triangle.*
(11) *Kiepert Center of the Excentral Triangle of the Fourth Brocard Triangle.*
(12) *Kiepert Center of the Hexyl Triangle of the Fourth Brocard Triangle.*
(13) *Perspector of the Fourth Brocard Triangle and the Triangle of the Circumcenters of the Triangulation Triangles of the Kiepert Center.*
(14) *Inverse of the Centroid in the Stevanovic Circle.*
(15) *Inverse of the Orthocenter in the Stevanovic Circle.*
(16) *Inverse of the Centroid in the Lester Circle.*
(17) *Inverse of the Orthocenter in the Lester Circle.*

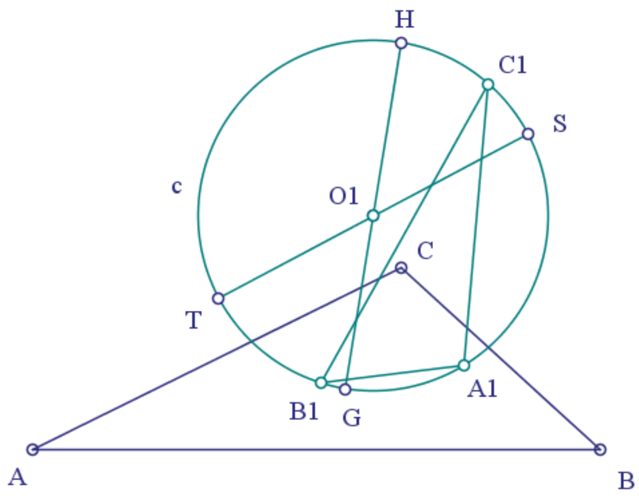

FIGURE 7.

Figure 7 illustrates the first two parts of Theorem 3.4 (See also [13]. D-triangle). In figure 7,

- G is the Centroid,
- H is the Orthocenter,
- c is the Orthocentroidal circle $C(2, 4)$,
- O_1 is the center of circle c,
- $A_1B_1C_1$ is the Fourth Brocard triangle (D-triangle in [13]),
- T is the Tarry point of triangle $A_1B_1C_1$,
- S is the Steiner point of triangle $A_1B_1C_1$.

We see that points T and S lie on the Circle $2,4$ because the Circle $2,4$ is the circumcircle of triangle $A_1B_1C_1$.

We have investigated 192 inverse images of notable points of triangle ABC with respect to circle C(2,4). Of these 18 points are Kimberling centers and the rest of 174 points are new points, not available in Kimberling [8]. See the Supplementary material.

Table 2 gives inverse images wrt circle C(2,4) which are Kimberling centers.

	Point P	**Inverse of Point P wrt circle C(2,4)**
1	X(3) Circumcenter	X(5)
2	X(5) Nine-Point Center	X(3)
3	X(6) Symmedian Point	X(115)
4	X(13) Outer Fermat Point	X(14)
5	X(14) Inner Fermat Point	X(13)
6	X(20) de Longchamps Point	X(3091)
7	X(21) Schiffler Point	X(2476)
8	X(22) Exeter Point	X(5133)
9	X(23) Far-Out Point	X(5169)
10	X(24) Perspector of the Kosnita Triangle and the Orthic Triangle	X(1594)
11	X(25) Product of the Orthocenter and the Symmedian Point	X(427)
12	X(26) Circumcenter of the Tangential Triangle	X(5576)
13	X(27) Quotient of the Orthocenter and the Spieker Center	X(469)
14	X(28) Quotient of the Clawson Point and the Spieker Center	X(5142)
15	X(29) Ceva Product of the Incenter and the Orthocenter	X(5125)
16	X(111) Parry Point	X(6032)
17	X(115) Kiepert Center	X(6)
18	X(384) Conway Point	X(5025)

Table 2.

Also, we could investigate properties of the new points listed in the Supplementary material. Consider, for example the Inverse of the Incenter wrt Circle $C(2,4)$.

Theorem 3.5. *The Inverse of the Incenter wrt Circle $C(2,4)$ lies on the following circles:*

1. *Circle passing through the Incenter, Inner Fermat Point and Outer Fermat Point.*
2. *Circle passing through the Incenter, Kiepert Center and Symmedian Point.*
3. *Circle passing through the Circumcenter, Incenter and Nine-Point Center.*
4. *Circle passing through the Bevan Point, Center of the Orthocentroidal Circle and Weill Point.*

(5) *Inverse Circle of the Brocard Circle of the Intouch Triangle in the Orthocentroidal Circle.*
(6) *Inverse Circle of the Lester Circle of the Intouch Triangle in the Orthocentroidal Circle.*
(7) *Inverse Circle of the Orthocentroidal Circle of the Excentral Triangle in the Orthocentroidal Circle.*
(8) *Inverse Circle of the Excentral Circle of the Intangents Triangle in the Orthocentroidal Circle.*
(9) *Inverse Circle of the Brocard Circle of the Hexyl Triangle in the Orthocentroidal Circle.*
(10) *Inverse Circle of the Lester Circle of the Hexyl Triangle in the Orthocentroidal Circle.*
(11) *Inverse Circle of the Orthocentroidal Circle of the Fuhrmann Triangle in the Orthocentroidal Circle.*

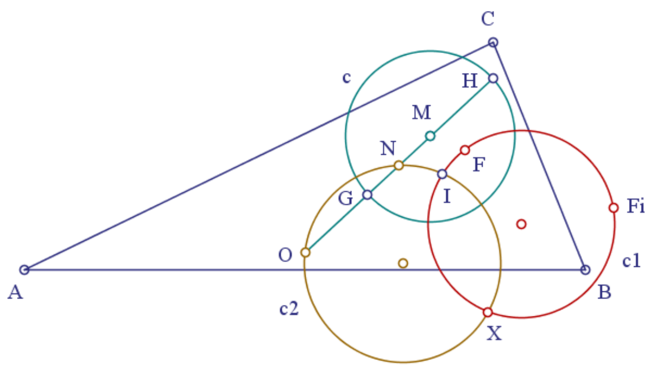

FIGURE 8.

Figure 8 illustrates parts (1) and (3) of Theorem . In figure 8,

- G is the Centroid,
- H is the Orthocenter,
- c is the Circle $C(2,4)$,
- I is the Incenter,
- F is the Outer Fermat Point,
- Fi is the Inner Fermat Point,
- c_1 is the circle through points I, F and Fi,
- O is the Cicumcenter,
- N is the Nine-Point Center,
- c_2 is the circle through I, N and O.
- X is the Inverse of I wrt circle c.

Point X lies on circles c_1 and c_2.

4. CIRCLE $C(2,5)$

The center of circle $C(2,5)$ is point X(54) and the radius is E.

We have investigated 193 inverse images of notable points of triangle ABC with respect to circle C(2,5). Of these 9 points are Kimberling centers and the rest of 184 points are new points, not available in Kimberling [8]. See the Supplementary material.

Table 3 gives inverse images wrt circle C(2,5) which are Kimberling centers.

	Point P	**Inverse of Point P wrt circle C(2,5)**
1	X(3) Circumcenter	X(1656)
2	X(4) Orthocenter	X(3090)
3	X(20) de Longchamps Point	X(7486)
4	X(21) Schiffler Point	X(7504)
5	X(22) Exeter Point	X(7571)
6	X(23) Far-Out Point	X(7570)
7	X(24) Perspector of the Kosnita Triangle and the Orthic Triangle	X(7569)
8	X(25) Product of the Orthocenter and the Symmedian Point	X(7539)
9	X(381) Center of the Orthocentroidal Circle	X(5055)

Table 3.

Also, we could investigate properties of the new points listed in the Supplementary material.

5. Circle $C(3,4)$

The center of circle $C(3,4)$ is point X(5) and the radius is $6E$.

We have investigated 192 inverse images of notable points of triangle ABC with respect to circle C(2,5). Of these 17 points are Kimberling centers and the rest of 175 points are new points, not available in Kimberling [8]. See the Supplementary material.

Table 4 gives inverse images wrt circle C(3,4) which are Kimberling centers.

Also, we could investigate properties of the new points listed in the Supplementary material.

6. Circle $C(3,5)$

The center of circle $C(3,5)$ is point X(140) and the radius is $3E$.

We have investigated 193 inverse images of notable points of triangle ABC with respect to circle C(2,5). Of these 12 points are Kimberling centers and the rest of 181 points are new points, not available in Kimberling [8]. See the Supplementary material.

Table 5 gives inverse images wrt circle C(3,5) which are Kimberling centers.

Also, we could investigate properties of the new points listed in the Supplementary material.

7. Circle $C(4,5)$

The center of circle $C(4,5)$ is point X(546) and the radius is $3E$.

We have investigated 194 inverse images of notable points of triangle ABC with respect to circle C(2,5). Of these 14 points are Kimberling centers and the rest of

	Point P	**Inverse of Point P wrt circle C(3,4)**
1	X(2) Centroid	X(20)
2	X(20) de Longchamps Point	X(2)
3	X(21) Schiffler Point	X(411)
4	X(22) Exeter Point	X(7503)
5	X(23) Far-Out Point	X(7527)
6	X(24) Perspector of the Kosnita Triangle and the Orthic Triangle	X(378)
7	X(25) Product of the Orthocenter and the Symmedian Point	X(1593)
8	X(26) Circumcenter of the Tangential Triangle	X(7526)
9	X(27) Quotient of the Orthocenter and the Spieker Center	X(7513)
10	X(28) Quotient of the Clawson Point and the Spieker Center	X(4219)
11	X(29) Ceva Product of the Incenter and the Orthocenter	X(412)
12	X(54) Kosnita Point	X(1141)
13	X(113) Jerabek Antipode	X(12162)
14	X(125) Center of the Jerabek Hyperbola	X(185)
15	X(381) Center of the Orthocentroidal Circle	X(382)
16	X(384) Conway Point	X(5999)
17	X(1141) Gibert Point	X(54)

TABLE 4.

179 points are new points, not available in Kimberling [8]. See the Supplementary material.

Table 6 gives inverse images wrt circle C(4,5) which are Kimberling centers.

Also, we could investigate properties of the new points listed in the Supplementary material.

8. SIMILITUDE CENTERS

Consider the circles from the set $\mathcal{C}$ and the similitude centers of pairs of circles from this set. There are 29 similitude centers of this kind. All they are Kimberling points. Table 7 lists a part of these points. For all points see the Supplementary material.

Figure 9 illustrates the first two rows in Table 7. In 9,

- c_1 is circle $C(2,3)$,
- c_2 is circle $C(2,4)$,
- the intersection point of circles c_1 and c_2 is the Centroid.This is the Internal center of similitude of circles c_1 and c_2,
- L is the de Longchamps point,

	Point P	Inverse of Point P wrt circle C(3,5)
1	X(2) Centroid	X(4)
2	X(4) Orthocenter	X(2)
3	X(20) de Longchamps Point	X(631)
4	X(21) Schiffler Point	X(6905)
5	X(22) Exeter Point	X(7509)
6	X(23) Far-Out Point	X(7550)
7	X(24) Perspector of the Kosnita Triangle and the Orthic Triangle	X(7503)
8	X(25) Product of the Orthocenter and the Symmedian Point	X(7395)
9	X(26) Circumcenter of the Tangential Triangle	X(7514)
10	X(28) Quotient of the Clawson Point and the Spieker Center	X(7549)
11	X(29) Ceva Product of the Incenter and the Orthocenter	X(7567)
12	X(381) Center of the Orthocentroidal Circle	X(1656)

Table 5.

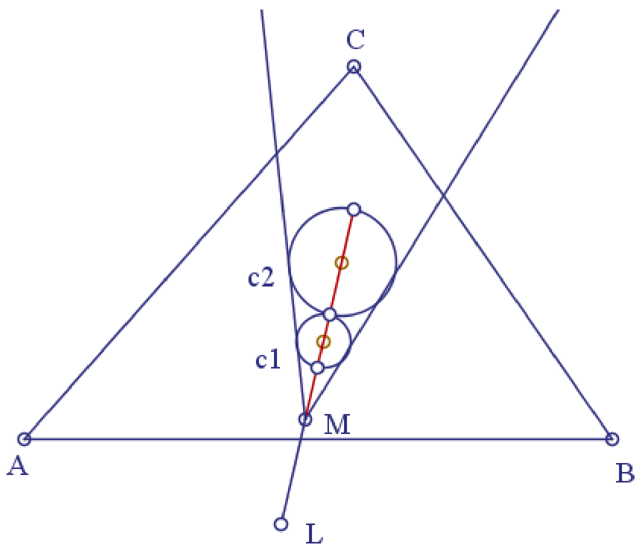

Figure 9.

- M is the External center of similitude of circles c_1 and c_2. At the same time point M is the midpoint of the Centroid and the de Longchamps point. This is point X(376) in [8].

We have investigated 56 similitude center from circle $C(2,4)$ to a number of named circles of triangle ABC. Of these 20 are Kimberling points and the rest of 36 are new points, not available in Kimberling [8]. The reader may see the results in the Supplementary material.

Also, we have investigated similitude centers from circles $C(2,3), C(2,5), C(3,4), C(3,5)$ and $C(4,5)$ to a number of named circle of triangle ABC. The investigated points

	Point P	**Inverse of Point P wrt circle C(4,5)**
1	X(2) Centroid	X(3091)
2	X(3) Circumcenter	X(381)
3	X(20) de Longchamps Point	X(3832)
4	X(21) Schiffler Point	X(7548)
5	X(22) Exeter Point	X(7566)
6	X(23) Far-Out Point	X(7565)
7	X(24) Perspector of the Kosnita Triangle and the Orthic Triangle	X(7547)
8	X(25) Product of the Orthocenter and the Symmedian Point	X(7507)
9	X(26) Circumcenter of the Tangential Triangle	X(7564)
10	X(27) Quotient of the Orthocenter and the Spieker Center	X(7563)
11	X(28) Quotient of the Clawson Point and the Spieker Center	X(7559)
12	X(29) Ceva Product of the Incenter and the Orthocenter	X(7541)
13	X(381) Center of the Orthocentroidal Circle	X(3)
14	X(389) Center of the Taylor Circle	X(7687)

TABLE 6.

are not Kimberling points, except for the similitude center in Theorem 8.1. See the Supplementary material.

Theorem 8.1. *The Internal Center of Similitude of the Circle* $C(3,4)$ *and the Nine-Point Circle is the point X(5).*

Proof. The circles are concentric with center the point X(5). □

	Role	**Kimberling point**
1	Internal Center of Similitude of Circle C23 and Circle C24	X(2)
2	External Center of Similitude of Circle C23 and Circle C24	X(376)
3	Internal Center of Similitude of Circle C23 and Circle C25	X(2)
4	External Center of Similitude of Circle C23 and Circle C25	X(381)
5	Internal Center of Similitude of Circle C23 and Circle C34	X(140)
6	External Center of Similitude of Circle C23 and Circle C34	X(3)
7	Internal Center of Similitude of Circle C23 and Circle C35	X(631)
8	External Center of Similitude of Circle C23 and Circle C35	X(3)
9	Internal Center of Similitude of Circle C23 and Circle C45	X(1656)
10	External Center of Similitude of Circle C23 and Circle C45	X(20)
11	Internal Center of Similitude of Circle C24 and Circle C25	X(5071)
12	External Center of Similitude of Circle C24 and Circle C25	X(2)
13	Internal Center of Similitude of Circle C24 and Circle C34	X(3091)
14	External Center of Similitude of Circle C24 and Circle C34	X(4)
15	Internal Center of Similitude of Circle C24 and Circle C35	X(3090)
16	External Center of Similitude of Circle C24 and Circle C35	X(20)

Table 7.

Supplementary Material

The enclosed supplementary material contains theorems related to the topic.

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