International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 2, 2017, pp.55-71.
Received 20 February 2017. Published on-line 28 February 2017
web: http://www.journal-1.eu/

Computer Discovered Mathematics: Half-Anticevian Triangle of the Incenter

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Abstract. By using the computer program "Discoverer" we study the Half-Anticevian triangle of the Incenter.

Keywords. Half-Anticevian triangle, triangle geometry, notable point, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

Let P be an arbitrary point in the plane of triangle ABC. Denote by $PaPbPc$ the Anticevian triangle of point P. Let Ma be the midpoint of segment APa and define Mb and Mc similarly. We call triangle $MaMbMc$ the *Half-Anticevian Triangle of Point P*.

Figure 1 illustrates the definition. In figure 1, P is an arbitrary point, $PaPbPc$ is the Anticevian triangle, and $MaMbMc$ is the Half-Anticevian Triangle of P.

Note that the Half-Anticevian Triangle of the Centroid is the Medial Triangle.

We encourage the students, teachers and researchers to investigate the Half-Anticevian triangles of other notable points, e.g. the Circumcenter, Orthocenter, Nine-Point Center and Symmedian Point, and to publish the results in our journal.

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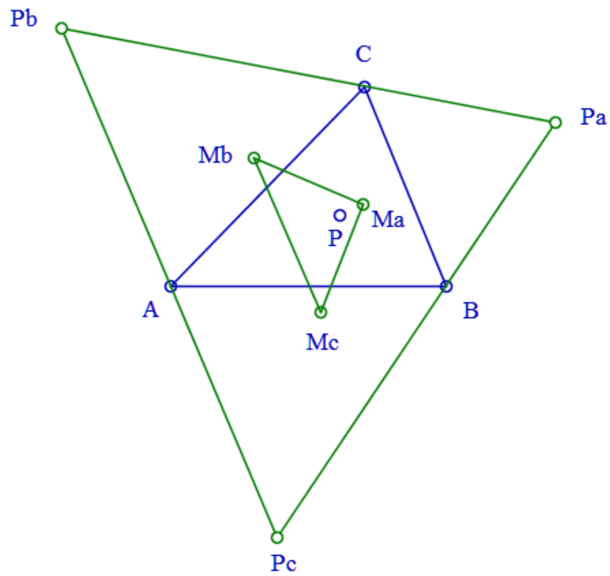

FIGURE 1.

Also, we encourage the readers to solve the problems posed in this paper and to publish the solutions in our journal.

We use the computer program ”Discoverer”, created by the authors.

We use barycentric coordinates. See [1]-[17].

Sometimes the barycentric coordinates of a point of the form $(f(a,b,c), f(b,c,a), f(c,a,b))$ are shortened to $[f(a,b,c)]$.

The area of the reference triangle ABC is denoted by Δ.

The Kimberling points [8] are denoted by $X(n)$.

2. HALF-ANTICEVIAN TRIANGLES OF POINT P

Theorem 2.1. *The barycentric coordinates of the Half-Anticevian triangle $MaMbMc$ of a point $P = (u, v, w)$ are as follows:*

(1) $Ma = (-2u+v+w, v, w),\ Mb = (u, u-2v+w, w),\ Mc = (u, v, u+v-2w).$

Proof. The vertices of the Anticevian triangle $PaPbOc$ of a point $P(u, v, w)$ has the following barycentric coordinates:

$$Pa = (-u, v, w),\ Pb = (u, -v, w),\ Pc = (u, v, -w).$$

By using formula (14),[5] we find the midpoint Ma of segment APa as follows: $Ma = (-2u+v+w, v, w)$. Similarly, we find the midpoints of segments BPb and CPc as follows: $Mb = (u, u-2v+w, w)$ and $Mc = (u, v, u+v-2w)$. □

Theorem 2.2. *The area of the Half-Anticevian triangle of point P is*

$$area = \frac{\Delta}{4}. \quad (2)$$

.

Proof. We use the area formula (2), [5]. □

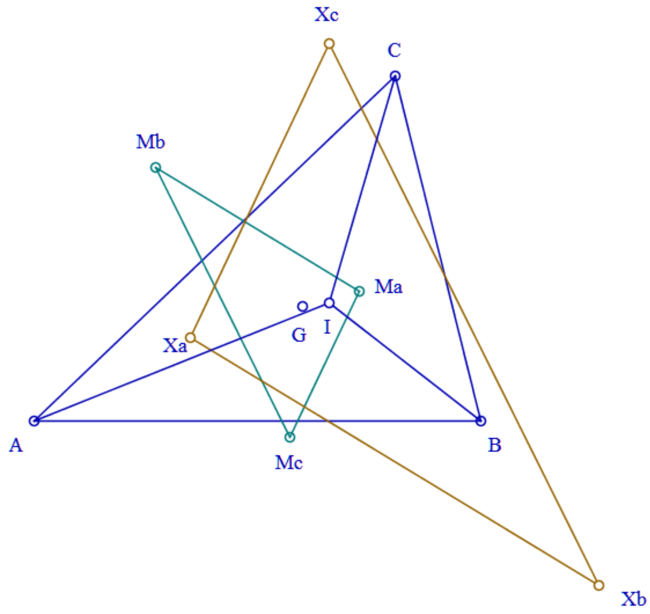

FIGURE 2.

3. SPECIAL CASE: P = INCENTER

3.1. **Barycentric Coordinates.** In this section we denote by $MaMbMc$ the Half-Anticevian triangle of the Incenter.

Theorem 3.1. *The barycentric coordinates of the Half-Anticevian triangle of the Incenter are as follows:*

(3) $Ma = (-2a+b+c, b, c), \quad Mb = (a, a-2b+c, c), \quad Mc = (a, b, a+b-2c).$

Proof. See 1. □

Theorem 3.2. *The side lengths of the Half-Anticevian triangle of the Incenter are as follows:*

$$a_I = MbMc = \frac{\sqrt{a(b-a+c)(2b^2+ab-4bc-a^2+ac+2c^2)}}{2\sqrt{(a-b+c)(a+b-c)}}, \tag{4}$$

$$b_I = McMa = \frac{\sqrt{b(a-b+c)(-b^2+ab+bc+2a^2-4ac+2c^2)}}{2\sqrt{(b-c+a)(b-a+c)}}, \tag{5}$$

$$c_I = MaMb = \frac{\sqrt{c(a+b-c)(2b^2+bc-4ab+2a^2+ac-c^2)}}{2\sqrt{(b-a+c)(a+b-c)}}. \tag{6}$$

Proof. We use the distance formula (9), [5]. □

3.2. **Homothetic Triangles.** We present triangles homothetic with the Half-Anticevian triangle of the Incenter.

Theorem 3.3. *The Half-Anticevian triangle of the Incenter and the Triangle of the Orthocenters of the Triangulation Triangles of the Incenter are homothetic. The Center of Homothety is the Centroid of triangle ABC and the ratio of the homothety is -2.*

Figure 2 illustrates Theorem 3.3. In fugure 2,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,
- I is the Incenter,
- Xa is the Orthocenter of triangle IBC,
- Xb is the Orthocenter of triangle ICA,
- Xc is the Orthocenter of triangle IAB,
- $XaXbXc$ is the Triangle of the Orthocenters of the Triangulation Triangles of the Incenter, and
- G is the Centroid.

Then G is the homothetic center of triangles $MaMbMc$ and $XaXbXc$.

Proof. We leave to the reader the proof that the triangles are homothetic. We will calculate the ratio of the homothety. By using the distance formula (9),[5], we find the side lengths of triangle IBC as follows:

$$a_T = BC = a, \tag{7}$$

$$b_t = CI = \sqrt{\frac{ab(a+b-c)}{a+b+c}}, \tag{8}$$

$$c_T = IB = \sqrt{\frac{ac(a-b+c)}{a+b+c}}. \tag{9}$$

By using the above side lengths, we find the barycentric coordinates of the orthocenter Xa of triangle IBC wrt triangle IBC. Then by using the change of coordinates formula (10),[5], we obtain the barycentric coordinates of point Xa wrt triangle ABC as follows:

$$Xa = (-a, a-c.a-b).$$

Now, by using the distance formula (9),[5] we calculate the distances GMa and GXa and then the ratio of homothety k as follows:

$$GMa = \frac{\sqrt{R}}{6\sqrt{b+c-a}},$$

$$GXa = \frac{-\sqrt{R}}{3\sqrt{b+c-a}},$$

where

$$R = 4a^3 + 2a^2c + 2a^2b - 8c^2a - 8b^2a + 9bca - b^2c + 2b^3 - bc^2 + 2c^3,$$

so that

$$k = \frac{-GXa}{GMa} = -2.$$

□

Theorem 3.4. *The Half-Anticevian triangle of the Incenter and the Pedal Triangle of the External Center of Similitude of the Incircle and the Circumcircle are homothetic. The Center of homothety is the point X(3911).*

Problem 3.1. *Find the ratio of the homothety in the previous theorem.*

Figure 3 illustrates Theorem 3.4. In figure 3, $MaMbMc$ is the Half-Anticevian triangle of the Incenter, Se is the External Similitude Center of Circumcircle and Incircle, and $XaXbXc$ is the Pedal triangle of point Se. The Center of the homothety $X(3911)$ is not in the figure.

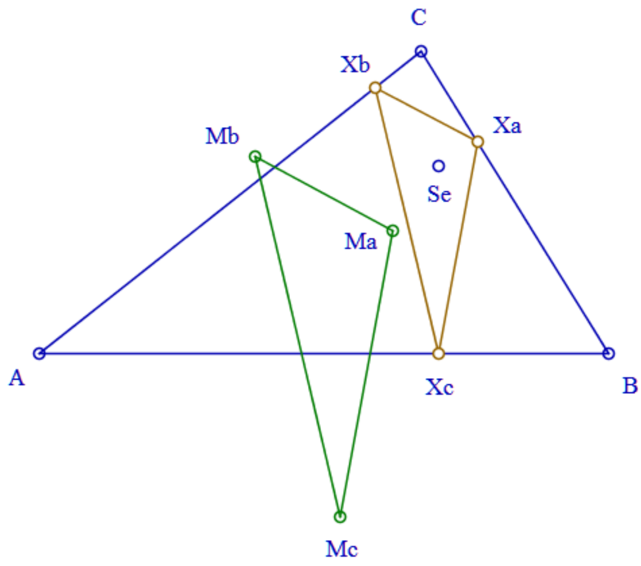

Figure 3.

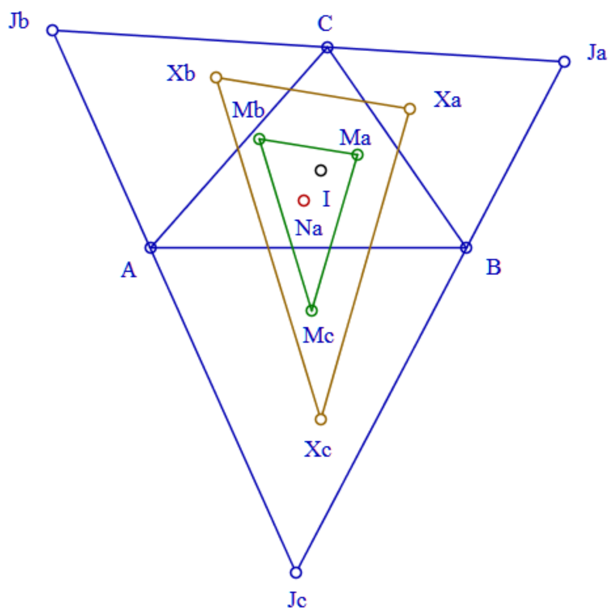

Figure 4.

Theorem 3.5. *The Half-Anticevian triangle of the Incenter and the Triangle of the de Longchamps Points of the Anticevian Corner Triangles of the Incenter are homothetic. The Center of the homothety is the X(8) Nagel point.*

Problem 3.2. *Find the ratio of the homothety in the previous theorem.*

Figure 4 illustrates Theorem 3.5. In figure 4,

- I is the Incenter,
- $JaJbJc$ is the Anticevian triangle of I, that is, the Excentral triangle,
- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,
- Xa is the de Longchamps point of triangle $AJbJc$,
- Xb is the de Longchamps point of triangle $BJcJa$,

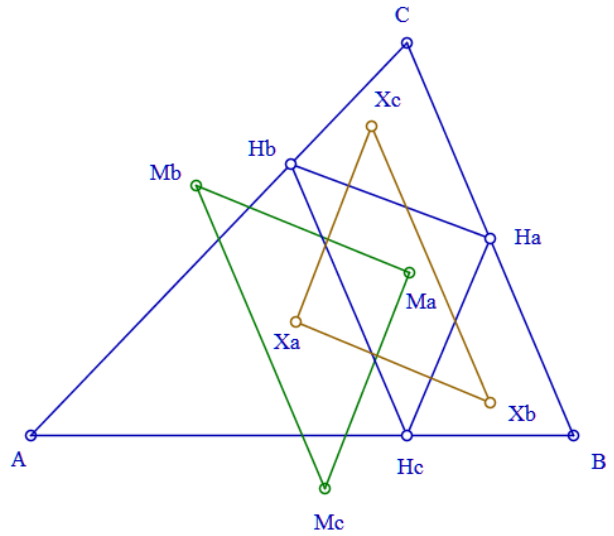

FIGURE 5.

- Xc is the de Longchamps point of triangle $CJaJb$, and
- $XaXbXc$ is the Triangle of the de Longchamps Points of the Anticevian Corner Triangles of the Incenter.

Then triangles $MaMbMc$ and $XaXbXc$ are homothetic and the Center of the homothety is the X(8) Nagel point.

Theorem 3.6. *The Half-Anticevian triangle of the Incenter and Triangle of the External Centers of Similitude of the Incircle and the Circumcircle of the Cevian Corner Triangles of the Orthocenter are homothetic.*

Problem 3.3. *Find the center and ratio of homothety in Theorem 3.6.*

Figure 5 illustrates Theorem 3.6. In figure 5,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,
- $HaHbHc$ is the Cevian triangle of the Orthocenter, that is, the Orthic triangle,
- Xa is the External Center of Similitude of the Incircle and Circumcircle of triangle $AHbHc$,
- Xb is the External Center of Similitude of the Incircle and Circumcircle of triangle $BHcHa$,
- Xc is the External Center of Similitude of the Incircle and Circumcircle of triangle $CHaHb$, and
- $XaXbXc$ is the Triangle of the External Centers of Similitude of the Incircle and the Circumcircle of the Cevian Corner Triangles of the Orthocenter.

Then triangles $MaMbMc$ and $XaXbXc$ are homothetic.

Theorem 3.7. *The Half-Anticevian triangle of the Incenter and the Antipedal Triangle of the Nagel Point are homothetic.*

Problem 3.4. *Find the center and ratio of homothety in Theorem 3.7.*

Figure 6 illustrates Theorem 3.7. In figure 6,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,

- $XaXbXc$ is the Antipedal Triangle of the Nagel Point, and
- X is the Center of the homothety of triangles $MaMbMb$ and $XaXbXc$.

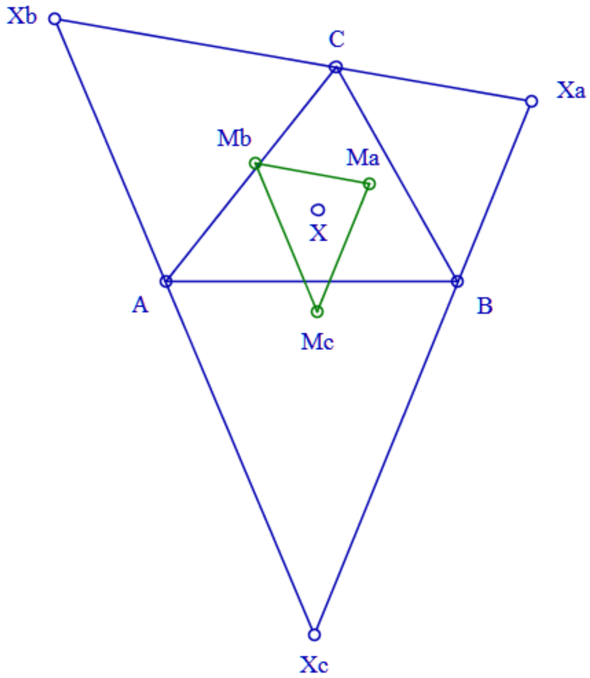

Figure 6.

Theorem 3.8. *The Half-Anticevian triangle of the Incenter and the Triangle of Reflections of the External Center of Similitude of the Incircle and the Circumcircle in the Sidelines of Triangle ABC are homothetic.*

Problem 3.5. *Find the center and ratio of homothety in Theorem 3.8.*

Figure 7 illustrates Theorem 3.8. In figure 7,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,
- Se is the External Center of Similitude of the Incircle and the Circumcircle,
- Xa is the Reflection of Se in the sideline BC,
- Xb is the Reflection of Se in the sideline CA,
- Xc is the Reflection of Se in the sideline AB,
- $XaXbXc$ is the Triangle of Reflections of the External Center of Similitude of the Incircle and the Circumcircle in the Sidelines of Triangle ABC, and
- X is the Center of the homothety of triangles $MaMbMb$ and $XaXbXc$.

Then triangles $MaMbMc$ and $XaXbXc$ are homothetic.

Theorem 3.9. *The Half-Anticevian triangle of the Incenter and the Triangle of the Circumcenters of the Triangulation Triangles of the Nagel Point are homothetic.*

Problem 3.6. *Find the center and ratio of homothety in Theorem 3.9.*

Figure 8 illustrates Theorem 3.9. In figure 8,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,

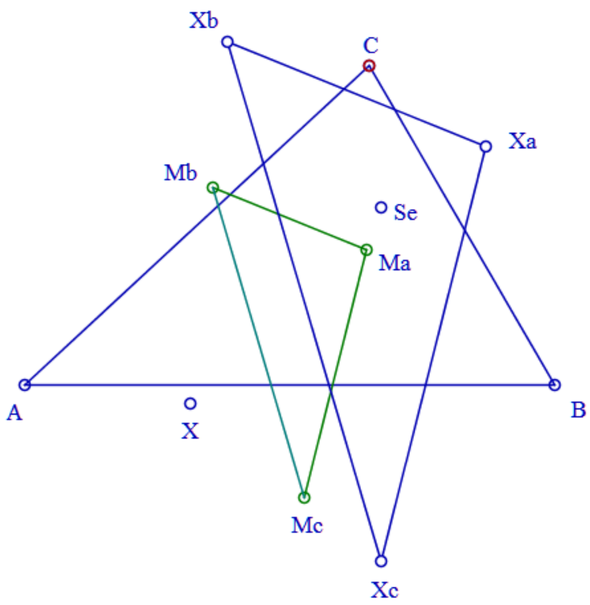

FIGURE 7.

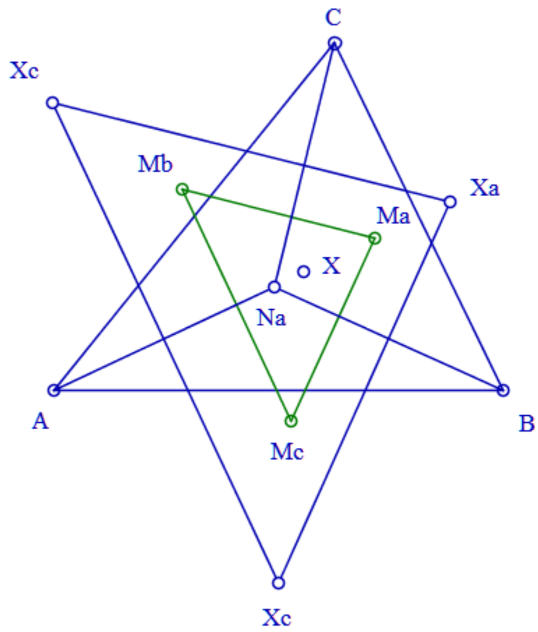

FIGURE 8.

- Na is the Nagel Point,
- Xa is the Circumcenter of triangle $NaBC$,
- Xb is the Circumcenter of triangle $NaCA$,
- Xc is the Circumcenter of triangle $NaAB$,
- $XaXbXc$ is the Triangle of the Circumcenters of the Triangulation Triangles of the Nagel Point, and
- X is the Center of homothety of triangles $MaMbMc$ and $XaXbXc$.

3.3. **Similar Triangles.**

Theorem 3.10. *The Half-Anticevuan triangle of the Incenter and the Circumcevian Triangle of the External Center of Similitude of the Incircle and the Circumcircle are similar (but not homothetic).*

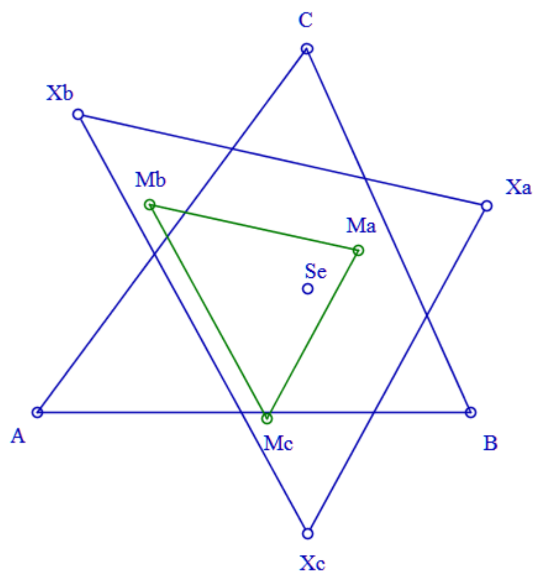

Figure 9.

Figure 9 illustrates Theorem 3.10. In figure 9,

- $MaMbMc$ is the Half-Anticevuan triangle of the Incenter,
- Se is the External Center of Similitude of the Incircle, and
- $XaXbXc$ is the Circumcevian Triangle of Se.

Proof. We will calculate the ratios

$$k_a = \frac{MbMc}{XbXc},\ k_b = \frac{McMa}{XcXa},\ k_c = \frac{MaMb}{XaXb}.$$

The side lengths of triangle $MaMbMc$ are known (see Theorem 3.2).

The barycentric coordinates of the Circumcevian triangle of the External Center of Similitude of the Incircle and the Circumcircle are as follows (see [3]):

$$Xa = \left(\frac{-a^2vw}{c^2v+b^2w}, v, w\right),\quad Xb = \left(u, \frac{-b^2wu}{a^2w+c^2u}, w\right),\quad Xc = \left(u, v, \frac{-c^2uv}{b^2u+a^2v}\right).$$

where u, v, w are the barycentric coordinates of the X(56) External Center of Similitude of the Incircle and Circumcircle, $Se = [a^2(a+b-c)(a-b+c)]$.

By using the distance formula (9),[5], we find the side lengths of triangle $XaXbXc$ as follows:

$$a_X = \frac{a(b+c-a)\sqrt{bc}}{\sqrt{(2b^2+bc-4ab+2a^2+ac-c^2)(-b^2+ab+bc+2c^2-4ac+2a^2)}},$$

$$b_X = \frac{b(a-b+c)\sqrt{ac}}{\sqrt{(2b^2+bc-4ab+2a^2+ac-c^2)(2b^2+ab-4bc+2c^2+ac-a^2)}},$$

$$c_X = \frac{c(a+b-c)\sqrt{ab}}{\sqrt{(2b^2+ab-4bc+2c^2+ac-a^2)(-b^2+ab+bc+2c^2-4ac+2a^2)}},$$

so that we see that

$$k_a = k_b = k_c.$$

□

Theorem 3.11. *The Half-Anticevian triangle of the Incenter and the Pedal Triangle of the Bevan-Schroder Point are similar.*

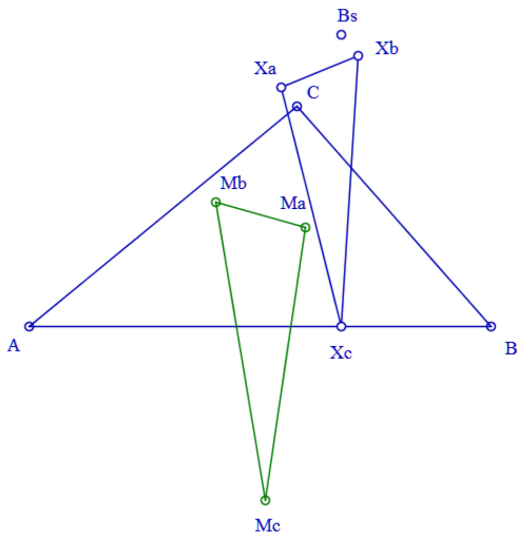

FIGURE 10.

Figure 10 illustrates Theorem 3.11. In figure 10,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,
- Bs is the Bevan-Schroder Point, and
- $XaXbXc$ is the Pedal Triangle of the Bevan-Schroder Point.

Problem 3.7. *Find the ratio of similitude in 3.11.*

Theorem 3.12. *The Half-Anticevian triangle of the Incenter and the Circumcevian Triangle of the Bevan-Schroder Point are similar.*

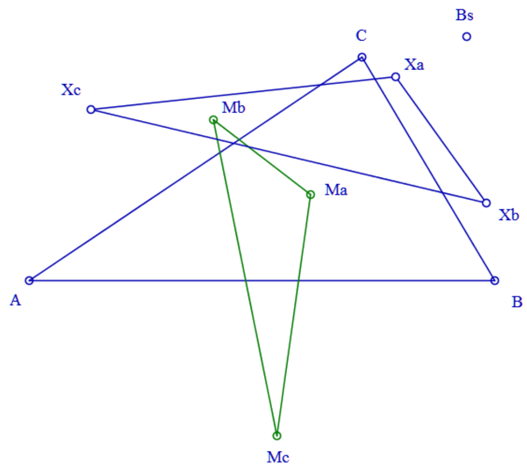

FIGURE 11.

Figure 11 illustrates Theorem 3.12. In figure 11,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,
- Bs ai the Bevan-Schroder Point, and
- $XaXbXc$ is the Circumcevian Triangle of the Bevan-Schroder Point.

Problem 3.8. *Find the ratio of similitude in 3.12.*

Theorem 3.13. *The Half-Anticevian triangle of the Incenter and the Triangle of Reflections of the Bevan-Schroder Point in the Sidelines of Triangle ABC are similar,*

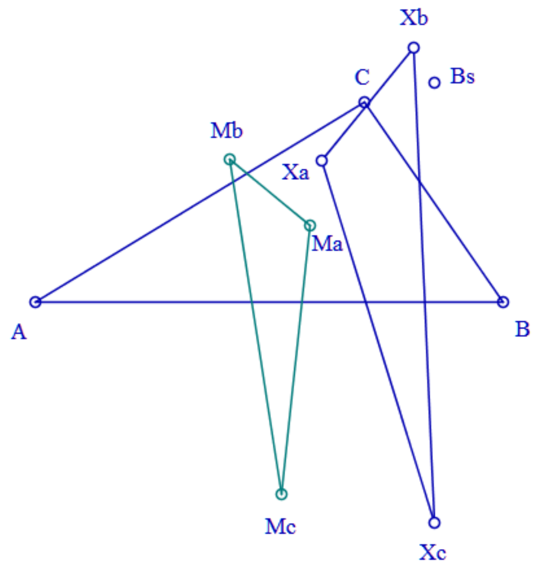

Figure 12.

Figure 12 illustrates Theorem 3.13. In figure 12,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,
- Bs is the Bevan-Schroder Point, and
- $XaXbXc$ is the Triangle of Reflections of the Bevan-Schroder Point in the Sidelines of Triangle ABC.

Problem 3.9. *Find the ratio of similitude in 3.13.*

Theorem 3.14. *The Half-Anticevian triangle of the Incenter and the Triangle of Reflections of the Nagel Point in the Sidelines of the Pedal Triangle of the Nagel Point are similar.*

Figure 13 illustrates Theorem 3.14. In figure 13,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,
- Na is the Nagel point,
- $PaPbPc$ is the Pedal Triangle of the Nagel Point,
- $XaXbXc$ is the Triangle of Reflections of the Nagel Point in the Sidelines of the Pedal Triangle of the Nagel Point.

Problem 3.10. *Find the ratio of similitude in 3.14.*

Theorem 3.15. *The Half-Anticevian triangle of the Incenter and the Triangle of Nine-Point Centers of the Triangulation Triangles of the Nagel Point are similar,*

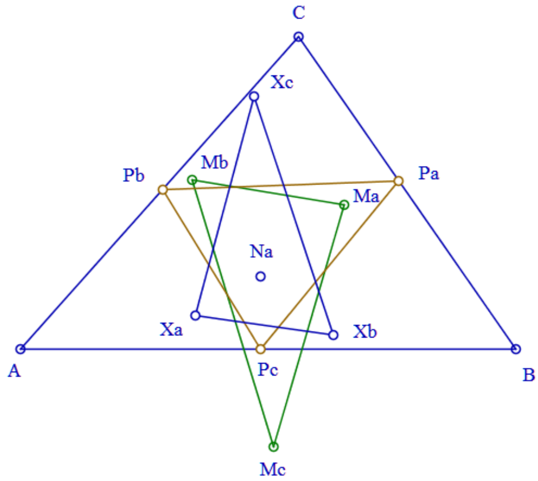

FIGURE 13.

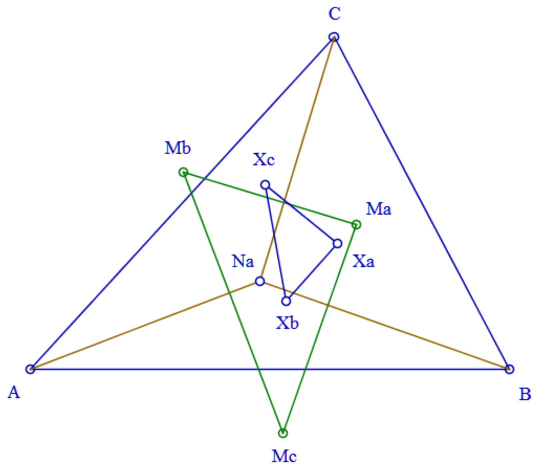

FIGURE 14.

Figure 14 illustrates Theorem 3.15. In figure 14,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,
- Na is the Nagel point,
- Xa is the Nine-Point Center of triangle $NaBC$,
- Xb is the Nine-Point Center of triangle $NaCA$,
- Xc is the Nine-Point Center of triangle $NaAB$, and
- $XaXbXc$ is the Triangle of Nine-Point Centers of the Triangulation Triangles of the Nagel Point.

Problem 3.11. *Find the ratio of similitude in 3.15.*

Theorem 3.16. *The Half-Anticevian triangle of the Incenter and the Triangle of the Bevan-Schroder Points of the Cevian Corner Triangles of the Orthocenter are similar.*

Figure 15 illustrates Theorem 3.16. In figure 15,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter,
- H is the Orthocenter,

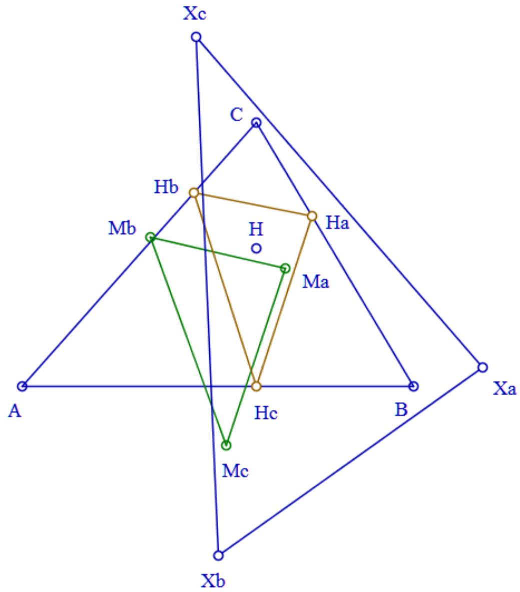

FIGURE 15.

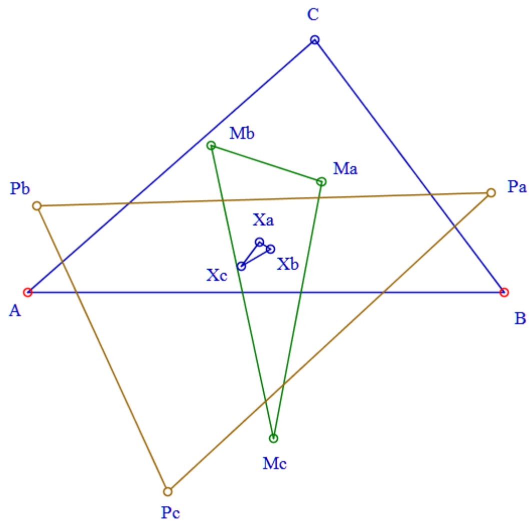

FIGURE 16.

- $HaHbHc$ is the Cevian triangle of the Orthocenter, that is, the Orthic triangle, Xa is the Bevan-Schroder Point of triangle $APbPc$, Xa is the Bevan-Schroder Point of triangle $BPcPa$, Xa is the Bevan-Schroder Point of triangle $CPaPb$, and $XaXbXc$ is the Triangle of the Bevan-Schroder Points of the Cevian Corner Triangles of the Orthocenter.

Problem 3.12. *Find the ratio of similitude in 3.16.*

Theorem 3.17. *The Half-Anticevian triangle of the Incenter and the Half-Circumcevian Triangle of the Nagel Point are similar.*

Figure 16 illustrates Theorem 3.17. In figure 16,

- $MaMbMc$ is the Half-Anticevian triangle of the Incenter, $PaPbPc$ is the Circumcevian triangle of the Nagel point,

- Xa is the midpoint of segment APa,
- Xb is the midpoint of segment BPb,
- Xc is the midpoint of segment CPc, and
- $XaXbXc$ is the Half-Circumcevian Triangle of the Nagel Point.

Problem 3.13. *Find the ratio of similitude in 3.17.*

3.4. **Kimberling Points.** We have investigated 195 notable points of the Half-Anticevian triangle of the Incenter. Of these 2 points are Kimberling centers and the rest of 193 points are new points.

Table 1 gives two notable points of the Half-Anticevian Triangle T of the Incenter in terms of the notable points of the Reference Triangle ABC that are Kimberling centers $X(n)$.

	Notable Points of triangle *T*	**Notable Points of Triangle *ABC***
1	Centroid	X(10164)
2	Circumcenter	X(11260)

TABLE 1.

3.5. **New Notable Points.** Below we give barycentric coordinates of a few new notable points. For a list of 193 new notable points see the Supplementary material.

Theorem 3.18. *The first barycentric coordinate of the Orthocenter of the Half-Anticevian triangle $MaMbMc$ of the Incenter is as follows:*

$$(b+c-a)(b+c-3a)(b^2+ab-2bc+ac+c^2).$$

Proof. We use the side lengths of triangle $MaMbMc$ in order to find the barycentric coordinates of the Orthocenter wrt triangle $MaMbMc$. Then we use the change of coordinates formula (10), [5] to find the barycentric coordinates of the Orthocenter wrt triangle ABC. □

Theorem 3.19. *The first barycentric coordinate of the Nine-Point Center of the Half-Anticevian triangle of the Incenter is as follows:*

$$2a^4+2a^3c+2a^3b-3a^2b^2-8a^2bc-3a^2c^2-2ab^3+4ab^2c+4abc^2-2c^3a-2b^2c^2+b^4+c^4.$$

Theorem 3.20. *The first barycentric coordinate of the Symmedian Point of the Half-Anticevian triangle of the Incenter is as follows:*

$$a(-7a^3c+a^2c^2+a^2b^2-7a^3b+7ab^3+6b^2c^2+7c^3a+20a^2bc$$
$$-13ab^2c-13abc^2+2a^4-3b^4-3c^4).$$

3.6. **Distances.**

Theorem 3.21. *The distance d between the Centroid of triangle ABC and Centroid of the Half-Anticevian triangle of the Incenter is*

$$d = \frac{\sqrt{(a+b+c)R_1}}{6\sqrt{(b+c-a)(c+a-b)(a+b-c)}}$$

where

$$R_1 = a^4 + c^4 + b^4 + ab^2c + a^2bc + ac^2b - cb^3 - ab^3 - a^3b - c^3b - a^3c - ac^3.$$

3.7. **Circumcircle.** Note that the barycentric coordinates of the Circumcenter of $MaMbMc$, that is $X(11260)$, given below, in [8] are still pending.

Theorem 3.22. *The first barycentric coordinate of the Circumcenter of the Half-Anticivian Triangle $MaMbMc$ of the Incenter is as follows:*

$$[a(2a^3 - a^2b - a^2c - 2ab^2 + 6acb - 2ac^2 + c^3 - 3bc^2 - 3b^2c + b^3)],$$

and the radius of the circumcircle of $MaMbMc$ is as follows:

$$R = \frac{\sqrt{abcR_1R_2R_3}}{2(b+c-a)(c+a-b)(a+b-c)\sqrt{a+b+c}}$$

where

$$R_1 = 4bc - a^2 + ab + ac + 2b^2 + 2c^2,$$
$$R_2 = 2a^2 - 4ac + ab - b^2 + bc + 2c^2,$$
$$R_3 = 2b^2 - 4ab + bc + 2a^2 + ac - c^2.$$

Proof. We use the side lengths of triangle $MaMbMc$ (see Theorem 3.2, in order to find the barycentric coordinates of the Circumcenter wrt triangle $MaMbMc$. Then we use formula (10), [5] and we obtain the barycentric coordinates of the Circumcenter wrt the Reference triangle ABC. By using formula (9), [5], we find the radius as the distance from the Circumcenter and the vertex Ma of triangle $MaMbMc$. □

Theorem 3.23. *The Inverse of the X(8) Nagel Point of Triangle ABC wrt the Circumcircle of the Half-Anticevian Triangle of the Incenter is the X(1319) Bevan-Schroder Point.*

Figure 17 illustrates Theorem 3.23. In figure 17, $MaMbMc$ is the Half-Anticevian triangle of the Incenter, c is the circumcircle of triangle $MaMbMc$, O is the circumcenter of triangle $MaMbMc$, Na is the Nagel point of triangle ABC, Bs is the Bevan-Schroder Point of triangle ABC. Then points Na and Bs are inverse points wrt circle c.

Proof. We use theorem 3.22 and the inversion formula (20), [5]. □

SUPPLEMENTARY MATERIAL

The enclosed supplementary material contains theorems related to the topic.

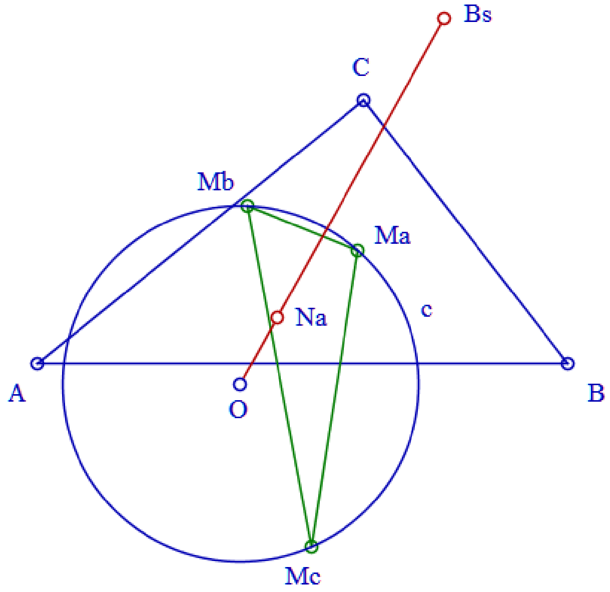

FIGURE 17.

Acknowledgement

The authors are grateful to Professor René Grothmann for his wonderful computer program *C.a.R.* http://car.rene-grothmann.de/doc_en/index.html. See also http://www.journal-1.eu/2016-1/Grothmann-CaR-pp.45-61.pdf.

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