A Note on the Leversha Point

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Abstract. We find the barycentric coordinates and new properties of the Leversha point in the triangle geometry.

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1. Introduction

In accordance with Leversha \cite{6} the internal center of similitude of the circumcircle of the Kosnita and tangential triangles is a significant triangle point. We call this point the Leversha point. Note that at present time the Leversha point is not included in the Kimberling’s ETC \cite{5}, the 11188 points edition of 2016. In this note we find the barycentric coordinates and new properties of the Leversha point.

Figures 1 and 2 illustrate the Leversha point. In figures 1 and 2, \(ABC\) is the reference triangle, \(OaObOc\) is the Kosnita triangle, \(c_1\) is the circumcircle of the Kosnita triangle, \(K\) is the circumcenter of the Kosnita triangle, \(TATBTC\) is the tangential triangle, \(c_2\) is the circumcircle of the tangential triangle, \(T\) is the circumcenter of the tangential triangle, and \(P\) is the internal center of similitude of circles \(c_1\) and \(c_2\).

We use barycentric coordinates. We refer the reader to \cite{11}, \cite{1}, \cite{5}, \cite{2}, \cite{3}, \cite{4}, \cite{7}, \cite{8}, \cite{9}, \cite{10}.

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Here we calculate the barycentric coordinates of the Kosnita triangle. The reader may find the definition of the Kosnita triangle e.g. in [3].

**Theorem 2.1.** The barycentric coordinates of the Kosnita triangle $OaObOc$ are as follows:

\[
Oa = (a^2(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2)),
\]
\[
-b^2(a^4 + b^4 - 2a^2b^2 - a^2c^2 - b^2c^2),
\]
\[
-c^2(a^4 + c^4 - 2a^2c^2 - a^2b^2 - b^2c^2)).
\]
Proof. The Kosnita triangle \( OaObOc \) and the tangential triangle \( TaTbTc \) are homothetic with center of homothety the circumcenter of triangle \( ABC \) and ratio 2 (see e.g. Laversha [11], Theorem 12.14). Hence, we can calculate the vertices \( Oa, Ob \) and \( Oc \) as midpoints of segments \( OTa, OTb \) and \( OTc \), by using the midpoint formula (14), [2]. The barycentric coordinate of the Circumcenter \( O \) and the Tangential triangle \( TaTbTc \) are given in [11], pages 26 and 54, respectively. Another way is to use the formula for homothety (17), [2], or to use the definition of the Kosnita triangle.

\[
Oa = \left( a^2(a^4 + b^4 - 2a^2b^2 - a^2c^2 - b^2c^2),ight.
\]
\[
-b^2(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2),
\]
\[
c^2(b^4 + c^4 - 2b^2c^2 - a^2b^2 - a^2c^2)).
\]
\[
Ob = \left( (a^2(a^4 + c^4 - 2a^2c^2 - a^2b^2 - b^2c^2),
\]
\[
b^2(b^4 + c^4 - 2b^2c^2 - a^2b^2 - a^2c^2),
\]
\[
-c^2(a^4 + b^4 + c^4 - 2a^2c^2 - 2b^2c^2)).
\]
\[
Oc = \left( a^2(a^4 + c^4 - 2a^2c^2 - a^2b^2 - b^2c^2),
\]
\[
b^2(b^4 + c^4 - 2b^2c^2 - a^2b^2 - a^2c^2),
\]
\[
-c^2(a^4 + b^4 + c^4 - 2a^2c^2 - 2b^2c^2)).
\]

Theorem 3.1. The barycentric coordinates of the Leversha Point \( P = (uP, bP, cP) \) are as follows:

\[
uP = a^2(3a^8 - 6a^6b^2 - 6a^6c^2 + 4b^2c^2a^4 - 2a^2b^2c^4 - 2a^2b^4c^2 + 6a^2b^6 + 6a^2c^6
\]
\[-3c^8 - 2b^4c^4 - 3b^8 + 4b^6c^2 + 4b^2c^6),\]
\[
vP = b^2(3b^8 - 6b^6c^2 - 6a^2b^6 + 4a^2b^4c^2 - 2b^2c^2a^4 - 2a^2b^2c^4 + 6b^6c^2 + 6a^2c^2
\]
\[-3a^8 - 2c^4a^4 - 3c^8 + 4a^2c^6 + 4a^2b^6),\]
\[
wP = c^2(3c^8 - 6a^2c^6 - 6b^2c^6 + 4a^2b^2c^4 - 2a^2b^4c^2 - 2b^2c^2a^4 + 6a^2c^2 + 6b^6c^2
\]
\[-3b^8 - 2a^2b^4 - 3a^8 + 4a^6b^2 + 4a^2b^6).\]

Proof. We use use the definition of the Leversha point and the barycentric coordinates of Kosnita triangle, given in Theorem 2.1.

The text two theorems and give alternative ways for finding the barycentric coordinates of the Leversha point. In order to find the barycentric coordinates of the Leversha point, we have to use the homothety formula (17), [2] in Theorem 3.2 and the internal division formula (12), [2] in Theorem 3.3.

Theorem 3.2. The Leversha Point is the Image of the Center of the Tangential Circle under the Homothety with Center at the Circumcenter and Ratio 2:3.

Theorem 3.3. The Leversha Point is the Point Dividing Internally the Directed Segment from the Circumcenter to the Circumcenter of the Tangential Triangle in the Ratio of 2:1.

Theorem 3.4. The Leversha Point lies on the Image of the Brocard Circle under the Homothety with Center the Center of the Tangential Circle and Ratio 1:3.

Theorem 3.5. The Leversha Point lies on the Image of the Lester Circle under the Homothety with Center the Center of the Tangential Circle and Ratio 1:3.
Figure 3.\[\text{Figure 3 illustrates Theorem 3.4. In figure 3, } T \text{ is the center of Tangential circle, } c_1 \text{ is the Brocard circle, } M \text{ is the center of the Brocard circle, } O \text{ is the circumcenter of triangle } ABC, \ c_2 \text{ is the image of the Brocard circle under homothety with center } T \text{ and ratio } 1:3, \ N \text{ is the center of circle } c_1, \text{ and } L \text{ is the Leversha point. Point } L \text{ lies on circle } c_2.\]

Figure 4.\[\text{Figure 4 illustrates Theorem 3.5. In figure 4, } T \text{ is the center of Tangential circle, } c_1 \text{ is the Lester circle, } M \text{ is the center of the Lester circle, } O \text{ is the circumcenter of triangle } ABC, \ c_2 \text{ is the image of the Lester circle under homothety with center } T \text{ and ratio } 1:3, \ N \text{ is the center of circle } c_1, \text{ and } L \text{ is the Leversha point. Point } L \text{ lies on circle } c_2.\]
We recommend the reader to generalize theorems 3.4 and 3.5 to the case where the circumcenter $O$ lies on an arbitrary circle.

REFERENCES


