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The Paskalev-Tchobanov Distance Formula and Some of its Applications

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Abstract. By using the Paskalev-Tchobanov distance formula, we give new proofs of a few well-known formulas from Euclidean geometry.

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The Paskalev-Tchobanov distance formula, published in 1985 (See §15, [2] and formula (9) in [1]), is a powerful tool for finding distances in triangle geometry.

By using the Paskalev-Tchobanov distance formula, we will give new proofs of a few well-known formulas form triangle geometry. There is a large number of formulas in triangle geometry which could be easily proved by the approcah of this paper.

We encourage the reader to extend the list of Theorems 2-7 given in this paper. .

Given triangle ABC with side-lengths $a = BC, b = CA, c = AB$. We denote by Δ the area of triangle ABC. Denote by

- $s = \dfrac{a+b+c}{2}$ the semiperimeter of triangle ABC.
- Δ the area of triangle ABC.
- r the radius of the Incircle.
- R the radius of the Circumcircle.

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- $m_a = AM_a,\ m_b = BM_b,\ m_c = CM_c$, the medians.
- $l_a = AL_a,\ l_b = BL_b,\ l_c = CL_c$, the internal bisectors.
- $h_a = AH_a,\ h_b = BH_b,\ h_c = CH_c$, the altitudes.

Denote by

- I the Incenter.
- O the Circumcenter.
- G_a projections of the Incenter on side BC.

The barycentric coordinates of vertices of triangle ABC are $A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1)$. The barycentric coordinates of the following points are well-known (See e.g. [1], [3]):

$$I = (a, b, c),\ O = (a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2)),$$
$$M_a = (0, 1, 1),\ M_b = (1, 0, 1),\ M_c = (1, 1, 0),$$
$$L_a = (0, b, c),\ L_b = (a, 0, c),\ L_c = (a, b, 0),$$
$$H_a = (0, (a^2 + b^2 - c^2)(b^2 + c^2 - a^2), (b^2 + c^2 - a^2)(c^2 + a^2 - b^2)),$$
$$G_a = (0, (b - c + a)(b + c - a), (c - a + b)(c + a - b)).$$

The Paskalev-Tcobanov distance theorem states:

Theorem 1. *Given two points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$ in normalized barycentric coordinates. Denote $x = u_1 - u_2, y = v_1 - v_2$ and $z = w_1 - w_2$. Then the square of the distance between P and Q is*

$$PQ^2 = -a^2yz - b^2zx - c^2xy.$$

Theorem 2. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$. *(Heron's Formula)*

Proof. By using Theorem 1, we find the length of the altitude

$$h_a = AH_a = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}.$$

Hence $\Delta = \dfrac{ah_a}{2} = \sqrt{s(s-a)(s-b)(s-c)}$ □

Theorem 3. $R = \dfrac{abc}{4\Delta}$.

Proof. By using Theorem 1, we obtain $R = AO = \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$, and by using Herons's formula we obtain $R = \dfrac{abc}{4\Delta}$, □

Theorem 4. $r = \dfrac{\Delta}{s}$

Proof. By using Theorem 1, we obtain $r = IG_a = \dfrac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$, and by using the Herons's formula we obtain $r = \dfrac{\Delta}{s}$. □

Theorem 5.

$$m_a = \frac{\sqrt{2b^2 + 2c^2 - a^2}}{2},\ m_b = \frac{\sqrt{2c^2 + 2a^2 - b^2}}{2},\ m_c = \frac{\sqrt{2a^2 + 2b^2 - c^2}}{2}.$$

Proof. By using Theorem 1, we obtain $m_a = AM_a = \dfrac{\sqrt{2b^2+2c^2-a^2}}{2}$. Similarly for m_b and m_c. □

Theorem 6. $l_a = \dfrac{\sqrt{bcs(s-a)}}{b+c},\ l_b = \dfrac{\sqrt{cas(s-b)}}{c+a},\ l_c = \dfrac{\sqrt{abs(s-c)}}{a+b}.$

Proof. By using Theorem 1, we obtain $l_a = AL_a = \dfrac{\sqrt{bcs(s-a)}}{b+c}$. Similarly for l_b and l_c. □

Theorem 7. $OI^2 = R(R-2r)$. *(Euler formula)*

Proof. By using Theorems 3 and 4 we obtain $R(R-2r) = \dfrac{abcE}{16\Delta^2}$, where

$$E = a^3 + b^3 + c^3 + 3abc - ba^2 - b^2a - ca^2 - c^2a - c^2b - b^2c.$$

By using Theorem 1, we obtain $OI^2 = \dfrac{abcE}{16\Delta^2}$. Hence $OI^2 = R(R-r)$. □

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