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# Problems for Students about Intouch Triangle

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**Abstract.** We present problems for students about triangles similar (but not homothetic) with the Intouch triangle. The problems are discovered by the computer program "Discoverer" created by the authors.

**Keywords.** Euclidean geometry; triangle geometry; computer discovered mathematics; "Discoverer".

**Mathematics Subject Classification (2010).** 51-04, 68T01, 68T99.

The Intouch triangle of a triangle $ABC$, also called the Contact triangle, is the triangle formed by the points of tangency of the incircle of triangle $ABC$ with triangle $ABC$. The Intouch triangle is also the Cevian triangle of triangle $ABC$ with respect to the Gergonne point. See also Contact triangle in [4].

We present problems for triangles similar (but not homothetic) with the Intouch triangle. The problems are discovered by the computer program "Discoverer" [1], [2], created by the authors. We encourage the students and teachers to solve the problems.

We denote the side lengths of triangle $ABC$ by $a = BC, b = CA$ and $c = AB$. Given triangles $PaPbPc$ and $QaQbQc$. The triangles are similar if and only if all the corresponding sides have lengths in the same ratio:

$$\frac{PbPc}{QbQc} = \frac{PcPa}{QcQa} = \frac{PaPb}{QaQb}$$



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We denote by $k = \dfrac{PbPc}{QbQc}$ the ratio of similarity of $PaPbPc$ to $QaQbQc$.

Reference for Problem 1: Fuhrmann triangle in [4].

**Problem 1.** *The Intouch triangle $PaPbPc$ is similar with triangle $QaQbQc$ = the Fuhrmann triangle. The ratio of similarity is*

$$k = \frac{(b+c-a)(c+a-b)(a+b-c)}{2\sqrt{abc}\sqrt{E}}$$

*where*

$$E = a^3 + b^3 + c^3 + 3abc - a^2b - a^2c - ab^2 - ac^2 - bc^2 - cb^2.$$

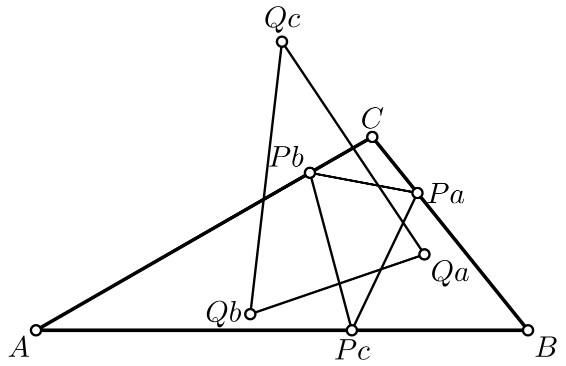


FIGURE 1.

Figure 1 illustrates Problem 1.

Reference for Problem 2: Pedal triangle in [4], Inversion in [4].

**Problem 2.** *The Intouch triangle $PaPbPc$ is similar with triangle $QaQbQc$ = the Pedal Triangle of the Inverse of the Incenter in the Circumcircle. The ratio of similarity is*

$$k = \frac{\sqrt{E}}{\sqrt{abc}}$$

*where $E$ is as in Problem 1*

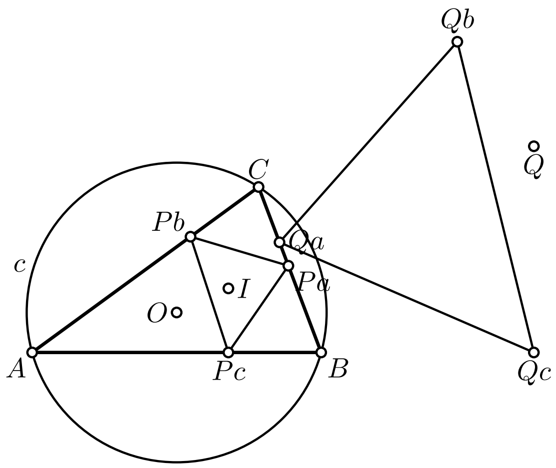


FIGURE 2.

Figure 2 illustrates Problem 2. In figure 2:

- $PaPbPc$ is the Intouch triangle,
- $I$ is the Incenter,
- $O$ is the circumcenter,
- $c$ is the circumcircle,
- $Q$ is the inverse point of the Incenter with respect to circcumcircle,
- $QaQbQc$ is the Pedal triangle of point $Q$.

Reference for Problem 3: Circumcevian triangle in [4], Inversion in [4].

**Problem 3.** *The Intouch triangle $PaPbPc$ is similar with triangle $QaQbQc$ = the Circumcevian Triangle of the Inverse of the Incenter in the Circumcircle. The ratio of similarity is*

$$k = \frac{(b+c-a)(c+a-b)(a+b-c)}{2abc}.$$

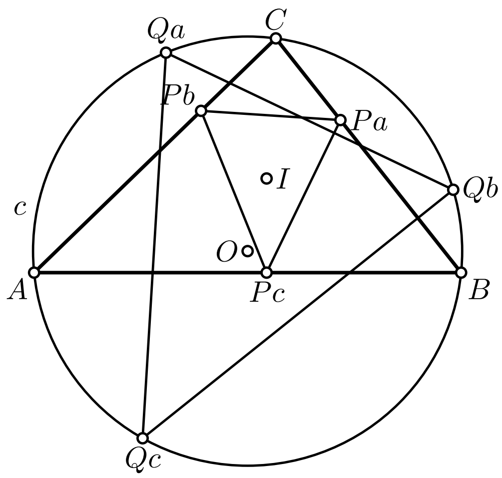


FIGURE 3.

Figure 3 illustrates Problem 3. In figure 3:

- $PaPbPc$ is the Intouch triangle,
- $I$ is the Incenter,
- $O$ is the circumcenter,
- $c$ is the circumcircle,
- $QaQbQc$ is the Circumcevian Triangle of the Inverse of the Incenter in the Circumcircle.

**Problem 4.** *The Intouch triangle $PaPbPc$ is similar with triangle $QaQbQc$ = the Triangle of Reflections of the Inverse of the Incenter in the Circumcircle in the Sidelines of Triangle $ABC$. The ratio of similarity is*

$$k = \frac{\sqrt{E}}{\sqrt{2abc}},$$

*where $E$ is as in Theorem 1.*

Figure 4 illustrates Problem 4. In figure 4:

- $PaPbPc$ is the Intouch triangle,
- $I$ is the Incenter,
- $O$ is the circumcenter,

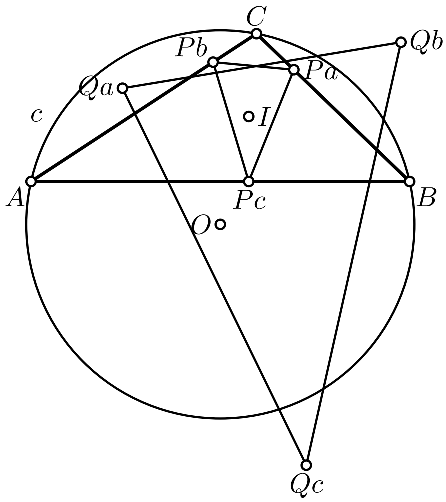


FIGURE 4.

- $c$ is the circumcircle,
- $QaQbQc$ is Triangle of Reflections of the Inverse of the Incenter in the Circumcircle in the Sidelines of Triangle ABC.

Let $RaRbRc$ be the Circumcevian triangle of a point $P$. Denote by $Qa$ the midpoint of segment $ARa$, by $Qb$ the midpoint of segment $BRb$,and by $Qc$ the midpoint of segment $CRc$. Then $QaQbQc$ is the Half-Circumcevian triangle of point $P$.

**Problem 5.** *The Intouch triangle $PaPbPc$ is similar with triangle $QaQbQc$ = the Half-Circumcevian Triangle of the Incenter. The ratio of similarity is*

$$k = \frac{(b+c-a)(c+a-b)(a+b-c)}{\sqrt{abcE}},$$

*where $E$ is as in Theorem 1.*

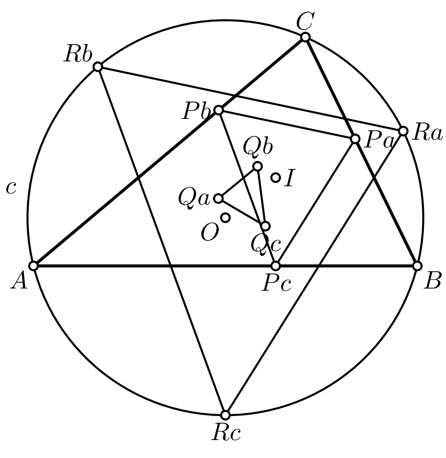


FIGURE 5.

Figure 5 illustrates Problem 5. In figure 5:

- $PaPbPc$ is the Intouch triangle,
- $I$ is the Incenter,
- $O$ is the circumcenter,

- $c$ is the circumcircle,
- $RaRbRc$ is the Circumcevian triangle of the Incenter,
- $Qa$ the midpoint of segment $ARa$,
- $Qb$ the midpoint of segment $BRb$,
- $Qc$ the midpoint of segment $CRc$,
- $QaQbQc$ is the Half-Circumcevian Triangle of the Incenter.

Reference for Problem 6: Nine-Point Center in [4].

**Problem 6.** *Denote by $I$ the Incenter of triangle $ABC$. Denote by $Qa$ the Nine-Point Center of triangle $IBC$, by $Qb$ the Nine-Point Center of triangle $AIC$, and by $Qc$ the Nine-Point Center of triangle $ABI$. The Intouch triangle $PaPbPc$ is similar with triangle $QaQbQc$ = the Triangle of the Nine-Point Centers of the Triangulation Triangles of the Incenter. The ratio of similarity is*

$$k = \frac{(b+c-a)(c+a-b)(a+b-c)}{\sqrt{abcE}},$$

*where $E$ is as in Theorem 1.*

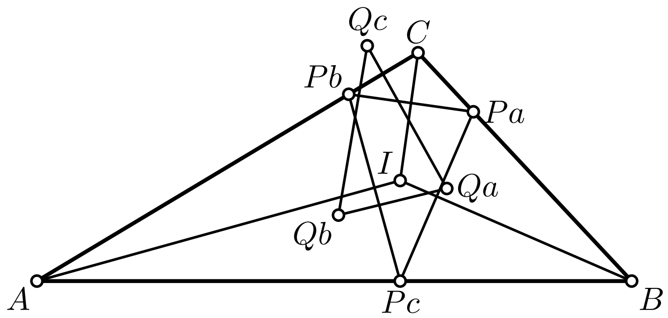


Figure 6.

Figure 6 illustrates Problem 6. In figure 6:

- $PaPbPc$ is the Intouch triangle,
- $I$ is the Incenter,
- $Qa$ is the Nine-Point Center of triangle $IBC$,
- $Qb$ is the Nine-Point Center of triangle $AIC$,
- $Qc$ is the Nine-Point Center of triangle $ABI$,
- $QaQbQc$ is the Triangle of the Nine-Point Centers of the Triangulation Triangles of the Incenter.

*Solution to Problem 6.* We use barycentric coordinates [3]. The Intouch triangle $PaPbPc$ is the Cevian triangle ([3], Section 8), of the Gergonne point ([3], Section 7) , hence it has barycentric coordinates

$$\begin{aligned} Pa &= (0, (b-c+a)(b+c-a), (c-a+b)(c+a-b)), \\ Pb &= ((a-b+c)(a+b-c), 0, (c-a+b)(c+a-b)), \\ Pc &= ((a-b+c)(a+b-c), (b-c+a)(b+c-a), 0). \end{aligned}$$

By using the distance formula (9) in [3], we find the lengths of segments form the Incenter $I = (a, b, c)$ to vertices $A = (1, 0, 0), B = (0, 1, 0)$ and $C = (0, 0, 1)$ as follows:

$$\begin{aligned} PA &= \sqrt{\frac{bc(b+c-a)}{(a+b+c)}}, \\ PB &= \sqrt{\frac{ac(a+c-b)}{(a+b+c)}}, \\ PC &= \sqrt{\frac{ab(a+b-c)}{(a+b+c)}}. \end{aligned}$$

Hence, the side lengths of triangle $T_1 = IBC$ are

$$a_1 = a, \quad b_1 = PC, \quad c_1 = PB,$$

the side lengths of triangle $T_2 = AIC$ are

$$a_2 = PC, \quad b_2 = b, \quad c_2 = PA,$$

and the side lengths of triangle $T_3 = ABI$ are

$$a_3 = PB, \quad b_3 = PA, \quad c_3 = c.$$

Now by using the change of coordinates formula (10) in [3] we obtain the barycentric coordinates wrt triangle $ABC$ of $Qa$ = Nine-Point Center ([3], Section 7) of triangle $IBC$. Similarly, we find the barycentric coordinates of $Qb$ and $Qc$. The barycentric coordinates are as follows:

$$\begin{aligned} Qa &= (a(b+c), b^2+2bc-a^2-ab+c^2, 2bc+c^2-a^2-ac+b^2), \\ Qb &= (-ab-b^2+2ac+c^2+a^2, b(a+c), 2ac+c^2+a^2-b^2-bc), \\ Qc &= (2ab+b^2-ac-c^2+a^2, a^2+2ab-bc-c^2+b^2, c(a+b)). \end{aligned}$$

Now by using the distance formula (9) in [3] we can calculate the lengths of sides of triangles $PaPbPc$ and $QaQbQc$, Denote

$$E = a^3+b^3+c^3+3abc-a^2b-a^2c-ab^2-ac^2-bc^2-cb^2.$$

. We obtain

$$\begin{aligned} PbPc &= \frac{(b+c-a)\sqrt{(c+a-b)(a+b-c)}}{2\sqrt{bc}}, \\ PcPa &= \frac{(c+a-b)\sqrt{(a+b-c)(b+c-a)}}{2\sqrt{ca}}, \\ PaPb &= \frac{(a+b-c)\sqrt{(b+c-a)(c+a-b)}}{2\sqrt{ab}}, \\ QbQc &= \frac{\sqrt{aE}}{2\sqrt{(c+a-b)(a+b-c)}}, \\ QcQa &= \frac{\sqrt{bE}}{2\sqrt{(a+b-c)(b+c-a)}}, \\ QaQb &= \frac{\sqrt{cE}}{2\sqrt{(b+c-a)(c+a-b)}}. \end{aligned}$$

Hence

$$\frac{PbPc}{QbQc} = \frac{PcPa}{QcQa} = \frac{PaPb}{QaQb} = \frac{(b+c-a)(c+a-b)(a+b-c)}{\sqrt{abcE}}.$$

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