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Inscribed triangles with centroid in a given point

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Abstract. Let P be a finite point in the plane. Let $\tau = A_P B_P C_P$ be a triangle with centroid P and vertices $A_P \in BC, B_P \in CA, C_P \in AB$. In this paper we study some properties of the triangle τ.

Keywords. Euclidean geometry, triangle geometry, barycentric coordinates, centroid, symmedian point.

1. Introduction

Let P be a finite point in the plane. Let $\tau = A_P B_P C_P$ be a triangle with centroid P and vertices $A_P \in BC, B_P \in CA, C_P \in AB$.

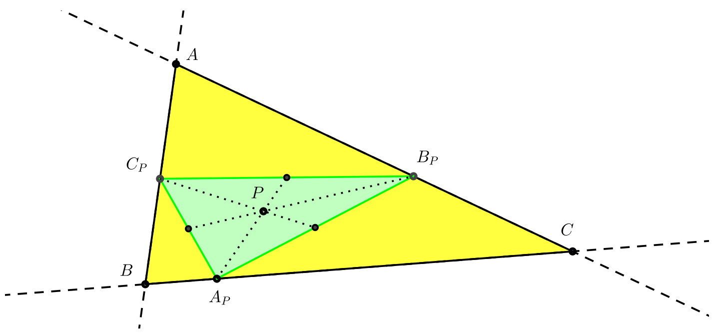

Figure 1. Triangle τ

Definition 1. [1, X(2)] **Centroid**

The centroid X_2 is the point of concurrence of the medians of ABC, situated 1/3 of the distance from each vertex to the midpoint of the opposite side.

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We shall work with homogeneous barycentric coordinates. We consider a nondegenerate triangle ABC as the reference triangle, and set up a coordinate system for points in the plane of the triangle.

$$A = (1 : 0 : 0)\ , \quad B = (0 : 1 : 0)\ , \quad C = (0 : 0 : 1)$$

We shall make use of John H. Conway's notations [3, §3.4.1]. Let S denote *twice* the area of triangle ABC. For a real number θ, denote $S_\theta = S\cot\theta$. In particular,

$$S_A = \frac{b^2+c^2-a^2}{2}\ , \quad S_B = \frac{c^2+a^2-b^2}{2}\ , \quad S_C = \frac{a^2+b^2-c^2}{2}$$

$$S_B + S_C = a^2\ , \quad S_C + S_A = b^2\ , \quad S_A + S_B = c^2$$

$$S_{AB} = S_A S_B\ , \quad S_{BC} = S_B S_C\ , \quad S_{CA} = S_C S_A$$

$$S^2 = S_{AB} + S_{BC} + S_{CA} = \frac{1}{4}(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)$$

2. Triangle τ

Let the point $P = (u : v : w)$. Let A_P is an arbitrary point on the sideline BC. $A_P = (0 : t : 1-t), t \in \mathbb{R}$. Let $B_P = (1-t_1 : 0 : t_1) \in CA, t_1 \in \mathbb{R}$ and $C_P = (t_2 : 1-t_2 : 0) \in AB, t_2 \in \mathbb{R}$, see Figure 1. The centroid of triangle $A_PB_PC_P$ is

$$G_t = \frac{1}{3}(A_P + B_P + C_P) = \frac{1}{3}(1-t_1+t_2)A + \frac{1}{3}(1+t-t_2)B + \frac{1}{3}(1-t+t_1)C$$

The point $G_t = P$ if and only if

$$\left|\begin{array}{l} \frac{1}{3}(1+t-t_2) = \dfrac{v}{u+v+w} \\ \frac{1}{3}(1-t+t_1) = \dfrac{w}{u+v+w} \end{array}\right.$$

Therefore

$$\left|\begin{array}{l} t_1 = \dfrac{-u-v+2w+t(u+v+w)}{u+v+w} \\ t_2 = \dfrac{u-2v+w+t(u+v+w)}{u+v+w} \end{array}\right.$$

Hence the vertexes of triangle τ are (we will use the matrix notation with vertex coordinates in the rows):

(1)

$$\tau = \begin{pmatrix} A_P \\ B_P \\ C_P \end{pmatrix} = \begin{pmatrix} 0 & t & 1-t \\ 2u+2v-w-t(u+v+w) & 0 & -u-v+2w+t(u+v+w) \\ u-2v+w+t(u+v+w) & 3v-t(u+v+w) & 0 \end{pmatrix}$$

Remark. In [2, (6)] is given more symmetric parametric representation[2] for τ:

$$\tau = \begin{pmatrix} 0 & \frac{1}{2}+V-W-T & \frac{1}{2}-V+W+T \\ \frac{1}{2}+U-W+T & 0 & \frac{1}{2}-U+W-T \\ \frac{1}{2}+U-V-T & \frac{1}{2}-U+V+T & 0 \end{pmatrix} \qquad (2)$$

(2) can be obtained from (1) with substitute t with $\frac{1}{2}+V-W-T$, and $\frac{u}{u+v+w}, \frac{v}{u+v+w}, \frac{w}{u+v+w}$ with U, V, W, respectively; $U+V+W=1$.

[2]In [2] Weise use the matrix notation with vertex coordinates in the columns.

3. Construction of the triangle

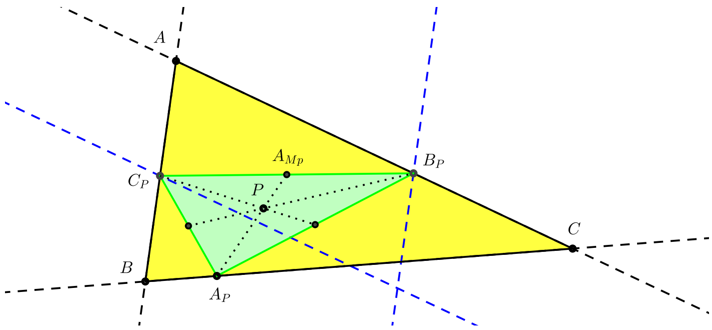

Figure 2. Construction

(1) Construct an arbitrary point A_P on the side-line BC.
(2) Point $A_{Mp} \in A_P P, \overrightarrow{A_P A_{Mp}} = \frac{3}{2}\overrightarrow{A_P P}$. This is the midpoint of $B_P C_P$.
(3) Lines b' and c' — reflections of CA and AB respectively in the point A_{Mp}.
(4) Points $B_P = c' \cap CA, C_P = b' \cap AB$. (see Figure 2).

4. Degenerate triangle τ

The triangle τ degenerates if, and only if, the three points A_P, B_P, C_P are collinear.

$$0 = \begin{vmatrix} 0 & t & 1-t \\ 2u+2v-w-t(u+v+w) & 0 & -u-v+2w+t(u+v+w) \\ u-2v+w+t(u+v+w) & 3v-t(u+v+w) & 0 \end{vmatrix}$$
$$= 3(t^2(u+v+w)^2 - t(u^2+4uv+3v^2+2vw-w^2) + v(2u+2v-w))$$

whence
(3)
$$t = \frac{u^2+4uv+3v^2+2vw-w^2 \pm (u+v+w)\sqrt{u^2+v^2+w^2-2(uv+vw+wu)}}{2(u^2+v^2+w^2+2uv+2vw+2wu)}$$

Number t is real if and only if $u^2+v^2+w^2-2uv-2vw-2wu \geqslant 0$. Last inequality represent Steiner in-ellipse with equation:

$$\varsigma_{(in)} : \quad x^2+y^2+z^2-2xy-2yz-2zx = 0$$

When point P is on, or outside the Steiner in-ellipse, exist the real number t, see (3), so that the triangle τ is degenerate. In this case the vertexes A_P, B_P, C_P are:

$$\begin{pmatrix} A_P \\ B_P \\ C_P \end{pmatrix} = \begin{pmatrix} 0 & u+3v-w \pm f & u-v+3w \mp f \\ 3u+v-w \mp f & 0 & -u+v+3w \pm f \\ 3u-v+w \pm f & -u+3v+w \mp f & 0 \end{pmatrix}$$
$$f = \sqrt{u^2+v^2+w^2-2(uv+vw+wu)}$$

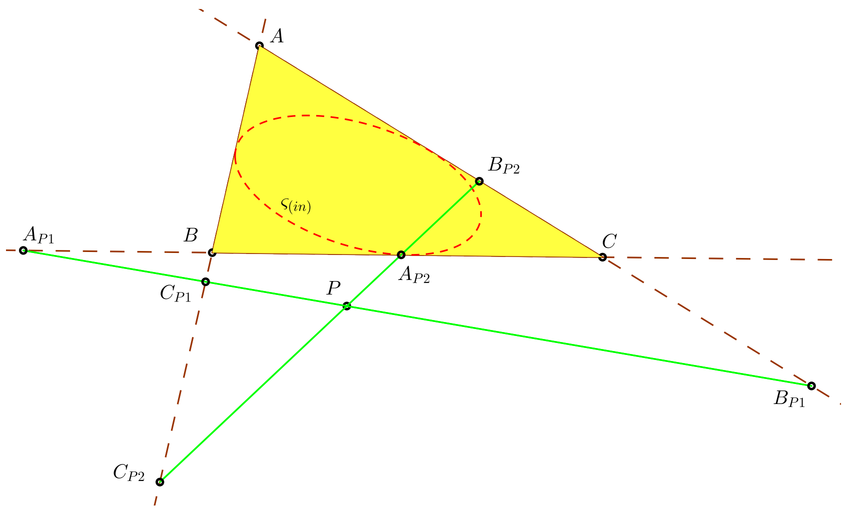

FIGURE 3. Degenerate triangles

5. ISOSCELES TRIANGLES

5.1. Square distance between vertexes.

Definition 2. [3, §7.1] **The distance formula in homogeneous barycentric coordinates**

If $P = (x : y : z)$ and $Q = (u : v : w)$, the **square distance** between P and Q is given by:

$$|PQ|^2 = \frac{1}{(u+v+w)^2(x+y+z)^2} \sum_{cyclic} S_A((v+w)x - u(y+z))^2$$

The square distances between B_P and C_P; C_P and A_P; A_P and B_P are given by:

$$\begin{aligned}
|B_PC_P|^2 &= \frac{1}{(u+v+w)^2}\Big(t^2(-a^2+2b^2+2c^2)(u+v+w)^2 \\
&+ t((a^2-3b^2-c^2)(u+4v-2w)+6(b^2-c^2)v)(u+v+w) \\
&+ b^2(u+v+w)^2 - 3(a^2-c^2)v(u+v+w) \\
&+ 9v(c^2v+a^2w-c^2w) + 3b^2(uv+v^2-2uw-4vw+w^2)\Big) \\
|C_PA_P|^2 &= \frac{1}{(u+v+w)^2}\Big(t^2(2a^2-b^2+2c^2)(u+v+w)^2 \\
&+ t(-3(a^2-b^2+3c^2)v(u+v+w)+2(-a^2+c^2)(u+v+w)^2) \\
&+ 9c^2v^2 + 3(a^2-b^2-c^2)v(u+v+w) + b^2(u+v+w)^2\Big) \\
|A_PB_P|^2 &= \frac{1}{(u+v+w)^2}\Big(t^2(2a^2+2b^2-c^2)(u+v+w)^2 \\
&- t(a^2+3b^2-c^2)(2u+2v-w)(u+v+w) + b^2(2u+2v-w)^2\Big)
\end{aligned} \tag{4}$$

5.2. Isosceles triangles τ_a with top vertex A_P.

$|C_PA_P| = |A_PB_P|$ is equivalent to

$$t = \frac{b^2u - S_Bv + b^2v - c^2v - S_Cw \mp f_a}{(b^2-c^2)(u+v+w)}$$

$$f_a = \sqrt{(b^2c^2u^2 + c^2a^2v^2 + a^2b^2w^2) - 2(a^2S_Avw + b^2S_Bwu + c^2S_Cuv) - S^2(v-w)^2}$$

For this two values of t correspond two, may be not real, isosceles triangles τ_{a1}, τ_{a2}:

$$(5)\quad \begin{aligned} A_{Pa1} &= (0 : b^2u - S_Bv + b^2v - c^2v - S_Cw - f_a : -c^2u + S_Bv + S_Cw + b^2w - c^2w + f_a) \\ B_{Pa1} &= (b^2u - 2c^2u + S_Cv + S_Bw + f_a : 0 : c^2u - S_Bv - S_Bw + b^2w - c^2w - f_a) \\ C_{Pa1} &= (2b^2u - c^2u - S_Cv - S_Bw - f_a : -b^2u + S_Cv + b^2v - c^2v + S_Cw + f_a : 0) \\ A_{Pa2} &= (0 : b^2u - S_Bv + b^2v - c^2v - S_Cw + f_a : -c^2u + S_Bv + S_Cw + b^2w - c^2w - f_a) \\ B_{Pa2} &= (b^2u - 2c^2u + S_Cv + S_Bw - f_a : 0 : c^2u - S_Bv - S_Bw + b^2w - c^2w + f_a) \\ C_{Pa2} &= (2b^2u - c^2u - S_Cv - S_Bw + f_a : -b^2u + S_Cv + b^2v - c^2v + S_Cw - f_a : 0) \\ f_a &= \sqrt{(b^2c^2u^2 + c^2a^2v^2 + a^2b^2w^2) - 2(a^2S_Avw + b^2S_Bwu + c^2S_Cuv) - S^2(v-w)^2} \end{aligned}$$

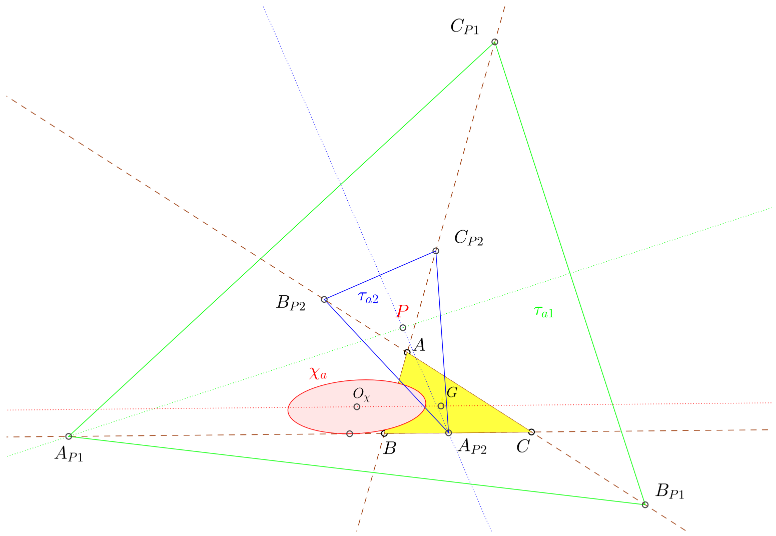

FIGURE 4. τ_{a1}, τ_{a2}

The triangles τ_{a1}, τ_{a2} are not real, if the point P is inside the conic χ_a, see Figure 4.

$$\chi_a : (b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2) - 2(a^2S_Ayz + b^2S_Bzx + c^2S_Cxy) - S^2(y-z)^2 = 0$$

The conic χ_a has discriminant $-48S^2(b^2-c^2)^2 \leqslant 0$ and χ_a is a ellipse, if $b \neq c$ (see [3, §10.7.1]). The ellipse χ_a has center at the point $O_\chi = (-2b^2 + 2c^2 : -a^2 - 3b^2 + c^2 : a^2 - b^2 + 3c^2), GO_\chi \| BC$, and χ_a tangent to BC at point $(0 : S_C : -S_B)$.

5.3. Isosceles triangles τ_b, τ_c with top vertexes B_P, C_P. .

$\underline{\tau_b}$:

$$(6)\quad \begin{aligned} A_{Pb1,2} &= (0 : S_Cu + a^2v - 2c^2v + S_Aw \pm f_b : -S_Au + c^2v - S_Aw + a^2w - c^2w \mp f_b) \\ B_{Pb1,2} &= (-S_Au + a^2u - c^2u + a^2v - S_Cw \mp f_b : 0 : S_Au - c^2v + S_Cw + a^2w - c^2w \pm f_b) \\ C_{Pb1,2} &= (S_Cu + a^2u - c^2u - a^2v + S_Cw \pm f_b : -S_Cu + 2a^2v - c^2v - S_Aw \mp f_b : 0) \\ f_b &= \sqrt{(b^2c^2u^2 + c^2a^2v^2 + a^2b^2w^2) - 2(a^2S_Avw + b^2S_Bwu + c^2S_Cuv) - S^2(w-u)^2} \end{aligned}$$

$\underline{\tau_c}$:

$$(7)\quad \begin{aligned} A_{Pc1,2} &= (0 : -S_Au - S_Av + a^2v - b^2v + b^2w \mp f_c : S_Bu + S_Av + a^2w - 2b^2w \pm f_c) \\ B_{Pc1,2} &= (S_Bu + a^2u - b^2u + S_Bv - a^2w \pm f_c : 0 : -S_Bu - S_Av + 2a^2w - b^2w \mp f_c) \\ C_{Pc1,2} &= (-S_Au + a^2u - b^2u - S_Bv + a^2w \mp f_c : S_Au + S_Bv + a^2v - b^2v - b^2w \pm f_c : 0) \\ f_c &= \sqrt{(b^2c^2u^2 + c^2a^2v^2 + a^2b^2w^2) - 2(a^2S_Avw + b^2S_Bwu + c^2S_Cuv) - S^2(u-v)^2} \end{aligned}$$

6. PERSPECTIVE TRIANGLES τ AND ABC

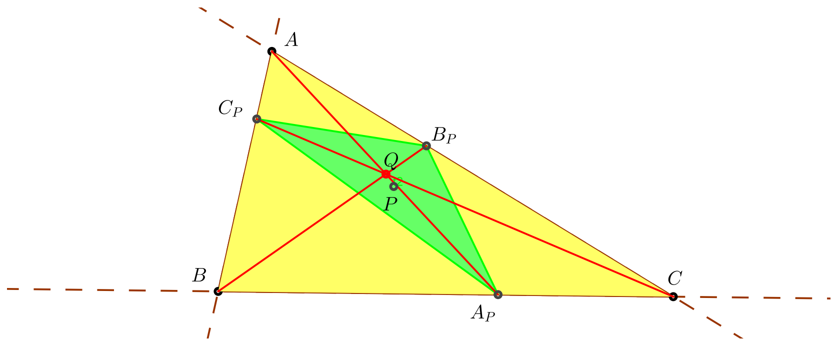

FIGURE 5. Perspective triangles τ and ABC

The equations of lines AA_P, BB_P, CC_P are:

$$AA_P : (-1+t)y + tz = 0$$
$$BB_P : ((-1+t)u + (-1+t)v + (2+t)w)x + ((-2+t)u + (-2+t)v + (1+t)w)z = 0$$
$$CC_P : (-3v + t(u+v+w))x + (u + tu - 2v + tv + w + tw)y = 0$$

The lines AA_P, BB_P, CC_P are concurrent at the point Q if

$$0 = \begin{vmatrix} 0 & t-1 & t \\ -u-v+2w+t(u+v+w) & 0 & -2u-2v+w+t(u+v+w) \\ -3v+t(u+v+w) & u-2v+w+t(u+v+w) & 0 \end{vmatrix}$$
$$= t^3(2u^2+4uv+2v^2+4uw+4vw+2w^2) + t^2(-3u^2-12uv-9v^2-6vw+3w^2)$$
$$+ t(u^2+14uv+13v^2+2uw-4vw+w^2) - 6uv - 6v^2 + 3vw$$

$$t = \frac{3^{1/3}f_4 + 3(u^2+4uv+3v^2+2vw-w^2)}{6(u+v+w)^2} + \frac{3^{2/3}(u^2+v^2-10vw+w^2-10u(v+w))}{6f_4}$$

$$f_4 = \Big(36(u-v)(u-w)(v-w)(u+v+w)^3$$
$$+ \sqrt{3}\sqrt{(u+v+w)^6(432(u-v)^2(u-w)^2(v-w)^2 - (u^2+v^2-10vw+w^2-10u(v+w))^3)}\Big)^{1/3}$$

The triangle τ is

$$A_P = (0 : \frac{3^{1/3}f_4 + 3(u^2+4uv+3v^2+2vw-w^2) + \frac{3^{2/3}(u+v+w)^2(u^2+v^2-10vw+w^2-10u(v+w))}{f_4}}{6(u+v+w)^2}$$
$$: 1 - \frac{3^{1/3}f_4 + 3(u^2+4uv+3v^2+2vw-w^2) + \frac{3^{2/3}(u+v+w)^2(u^2+v^2-10vw+w^2-10u(v+w))}{f_4}}{6(u+v+w)^2})$$

$$B_P = (-\frac{1}{6f_4(u+v+w)}((3^{1/3}{f_4}^2 - 3f_4(3u^2+v^2-w^2+2u(2v+w)))$$
$$+ 3^{2/3}(u+v+w)^2(u^2+v^2-10vw+w^2-10u(v+w))) : 0$$
$$: \frac{1}{6f_4(u+v+w)}(3^{1/3}{f_4}^2 + 3f_4(-u^2+v^2+2uw+4vw+3w^2)$$
$$+ 3^{2/3}(u+v+w)^2(u^2+v^2-10vw+w^2-10u(v+w))))$$

$$C_P = (\frac{1}{6f_4(u+v+w)}(3^{1/3}{f_4}^2 + 3f_4(3u^2-v^2+w^2+2u(v+2w))$$
$$+ 3^{2/3}(u+v+w)^2(u^2+v^2-10vw+w^2-10u(v+w)))$$
$$: 3v - \frac{3^{1/3}f_4 + 3(u^2+4uv+3v^2+2vw-w^2) + \frac{3^{2/3}(u+v+w)^2(u^2+v^2-10vw+w^2-10u(v+w))}{f_4}}{6(u+v+w)} : 0)$$

If the point $P = G$, then $Q = P = G$.

7. SYMMEDIAN POINT OF τ

The symmedian point K_τ of the triangle τ has homogeneous barycentric coordinates with respect to triangle τ (see also (4)):

$$\begin{aligned}
K_\tau &= (|B_PC_P|^2 : |C_PA_P|^2 : |A_PB_P|^2)\\
&= \big(b^2((1-3t+2t^2)u^2 + u((5-9t+4t^2)v + (-4+3t+4t^2)w)\\
&+ 2((2-3t+t^2)v^2 + (-5+2t^2)vw + (2+3t+t^2)w^2)) - (-3v + t(u+v+w))\\
&(a^2((-1+t)u + (-1+t)v + (2+t)w) + c^2(u - 2tu - 2((-2+t)v + (1+t)w)))\\
&: -b^2(-1+t)(u+v+w)(u+tu-2v+tv+w+tw) + (-3v + 2t(u+v+w))\\
&(a^2(-1+t)(u+v+w) + c^2(u+tu-2v+tv+w+tw))\\
&: t(u+v+w)(a^2(2(-1+t)u + 2(-1+t)v + w + 2tw) - c^2((-2+t)u + (-2+t)v + (1+t)w))\\
&+ b^2(2(2-3t+t^2)u^2 + 2(2-3t+t^2)v^2 + (-4-3t+4t^2)vw\\
&+ (1+3t+2t^2)w^2 + u(4(2-3t+t^2)v + (-4-3t+4t^2)w))\big)
\end{aligned}$$

With respect to triangle ABC, the symmedian point of triangle τ has homogeneous barycentric coordinates:

(8)

$$\begin{aligned}
K_\tau &= |B_PC_P|^2A_P + |C_PA_P|^2B_P + |A_PB_P|^2C_P\\
K_\tau &= \big(-c^2(-t^2(u+5v-2w)(u+v+w)^2 + t^3(u+v+w)^3\\
&+ t(-2u^3 + 8v^3 + uv(11v-4w) + u^2(v-3w) + 2v^2w - 5vw^2 + w^3)\\
&+ v(2u^2 + u(-2v+w) - (-2v+w)^2)) + a^2(u+v+w)(v(2u+2v-w)\\
&+ t^2(2u^2 + 3u(v+w) + (v+w)^2) - t(2u^2 + 3v^2 + 2vw - w^2 + u(5v+w)))\\
&+ b^2((2-t-2t^2+t^3)u^3 + ((-1+t)v + tw)((-2+t)v + (1+t)w)^2\\
&+ u(3(-2+t)(-1+t)^2v^2 + (5-2t-10t^2+6t^3)vw + (-1-2t+2t^2+3t^3)w^2)+\\
&u^2(t(6v-4w) + w + 3t^3(v+w) - t^2(9v+2w)))\\
&: -t(u+v+w)(-3v + t(u+v+w))(-c^2((-1+t)u - 2v + tv + w + tw)\\
&+ a^2((-1+t)u - v + tv + w + tw)) + b^2(v(-2u-2v+w)^2 + t^2(u+v+w)^2(u+2v+w)\\
&- t(u^3 + 6v^3 + 5v^2w - 2vw^2 - w^3 + u^2(8v+w) + u(13v^2 + 6vw - w^2)))\\
&: -b^2(-1+t)(u+v+w)((u+v-2w)(2v-w) + t^2(u+v+w)^2\\
&- t(u^2 + 4uv + 3v^2 - 2uw - 3w^2)) + a^2(-1+t)(u+v+w)(2v(u+v-2w)\\
&+ t^2(u+v+w)^2 - t(u^2 + 4uv + 3v^2 - uw + vw - 2w^2))\\
&+ c^2(t^2(u+v+w)^2(u+v+2w) + v(2u^2 + 4uv + 2v^2 - 2uw + 7vw - 4w^2)\\
&- t(u^3 + 5u^2v + 3v^3 + 10v^2w + 5vw^2 - 2w^3 + u(7v^2 + 10vw - 3w^2)))\big)
\end{aligned}$$

7.1. **Locus of the symmedian point K_τ, when $P = G$.**

When $P = G$, the point K_τ (8) obtain following form:

$$\begin{aligned}
K_\tau =&(a^2(1-2t)^2 + (-1+t)t(c^2(1-3t) + b^2(-2+3t))\\
&: b^2(1-2t)^2 + (-1+t)t(a^2(1-3t) + c^2(-2+3t))\\
&: c^2(1-2t)^2 + (-1+t)t(b^2(1-3t) + a^2(-2+3t)))
\end{aligned}$$

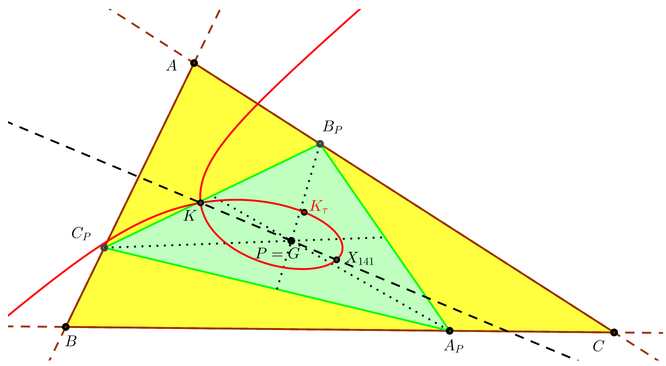

FIGURE 6. Locus of K_τ, if $P = G$

t	K_τ	barycentric coordinates
0	K	$(a^2 : b^2 : c^2)$
1	K	$(a^2 : b^2 : c^2)$
$\frac{1}{2}$	X_{141}	$(b^2 + c^2 : c^2 + a^2 : a^2 + b^2)$
∞	X_{523}	$(b^2 - c^2 : c^2 - a^2 : a^2 - b^2)$

7.2. **Locus of the symmedian point K_τ, when $P = K$.** When $P = K$, the point K_τ (8) obtain following form:

$$\begin{aligned} K_\tau =&\big((2a^2 + 2b^2 - c^2)(a^2 - b^2 + c^2) + (b^2 - c^2)(a^2 + b^2 + c^2)t \\ &: b^2(2a^2 + 2b^2 - c^2) + (c^2 - a^2)(a^2 + b^2 + c^2)t \\ &: -a^4 + b^4 + a^2c^2 + 2c^4 + (a^2 - b^2)(a^2 + b^2 + c^2)t\big) \end{aligned}$$

The locus of the points K_τ is represented by the straight line $\mathcal{L}$, see Figure 7:

$$\begin{aligned} \mathcal{L}\colon\ &(-a^6 - 2b^6 - 2c^6 + 2a^4(b^2 + c^2) + a^2(b^4 + c^4) - a^2b^2c^2)x \\ &+ (-2a^6 - b^6 - 2c^6 + 2b^4(c^2 + a^2) + b^2(c^4 + a^4) - a^2b^2c^2)y \\ &+ (-2a^6 - 2b^6 - c^6 + 2c^4(a^2 + b^2) + c^2(a^4 + b^4) - a^2b^2c^2)z = 0 \end{aligned}$$

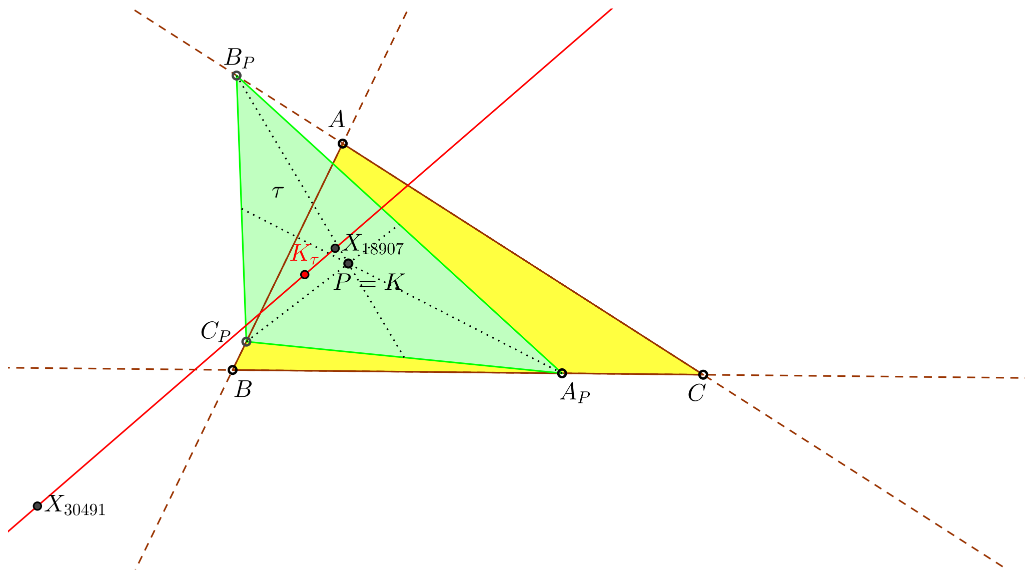

FIGURE 7. Locus of K_τ, if $P = K$

Points on the line $\mathcal{L}$, see [1]:

Point K_τ	The first barycentric coordinate of K_τ
X_{523}	b^2-c^2, point at infinity
X_{18907}	$4a^4+a^2(b^2+c^2)-(b^2-c^2)^2$
X_{30489}	$a^2(b^2+c^2)(2a^2+2b^2-c^2)(2a^2+2c^2-b^2)$
X_{30491}	$a^2(b^2-c^2)(-a^2+b^2+c^2)(2a^2+2b^2-c^2)(2a^2+2c^2-b^2)$

$$\begin{aligned}
\mathcal{L}\cap OG &= (4a^{10}-4a^6(b^2-c^2)^2-3a^2b^2c^2(b^2-c^2)^2-3a^8(b^2+c^2)-(b^2-c^2)^2(b^6-3b^4c^2-3b^2c^4+c^6)\\
&\quad +a^4(4b^6-3b^4c^2-3b^2c^4+4c^6):\cdots:\cdots)\\
\mathcal{L}\cap OI &= (a(-b^9-4a^7bc+b^8c+4b^7c^2-4b^6c^3-4b^3c^6+4b^2c^7+bc^8-c^9+2a^8(b+c)+a^5bc(b^2+c^2)\\
&\quad -3abc(b^2-c^2)^2(b^2+c^2)+a^6(-3b^3+2b^2c+2bc^2-3c^3)+2a^3bc(b^4-b^2c^2+c^4)\\
&\quad -a^4(b^5+b^4c-b^3c^2-b^2c^3+bc^4+c^5)+3a^2(b^7-b^5c^2-b^2c^5+c^7)):\cdots:\cdots)\\
\mathcal{L}\cap GI &= (4a^7-2a^6(b+c)-a^5(b^2+c^2)+3a(b^2-c^2)^2(b^2+c^2)+a^4(b^3+c^3)-2a^3(b^4-b^2c^2+c^4)\\
&\quad -(b-c)^2(b^5+4b^4c+5b^3c^2+5b^2c^3+4bc^4+c^5)+a^2(2b^5-b^3c^2-b^2c^3+2c^5):\cdots:\cdots)\\
\mathcal{L}\cap GK &= (4a^8-a^4(b^2-c^2)^2-3a^6(b^2+c^2)-(b^4-c^4)^2+a^2(5b^6-4b^4c^2-4b^2c^4+5c^6):\cdots:\cdots)\\
\mathcal{L}\cap KI &= (a(-2a^7(b+c)+2a^6(b^2+c^2)+a^5(b^3+c^3)+(b^4-c^4)^2-a^4(b^4+c^4)\\
&\quad -a(b-c)^2(b^5+4b^4c+5b^3c^2+5b^2c^3+4bc^4+c^5)+a^3(2b^5-b^3c^2-b^2c^3+2c^5)\\
&\quad +a^2(-2b^6+b^4c^2+b^2c^4-2c^6)):\cdots:\cdots)
\end{aligned}$$

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