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EULER'S LINE, EULER'S CURVE AND THÉBAULT'S POINT

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Abstract: Generalizations of Euler's line and Euler's circle of a triangle are considered. A Thébault's theorem is generalized by using them.
Key words: triangle, circumscribed conic section, parabola, ellipse, hyperbola, orthocenter, Euler's circle, Euler's line, Thebault's point

INTRODUCTION

There is a variety of assertions which allow to obtain more general and nice facts by changing some basic parameters. In some cases the new assertion turns out to be a generalization of a part of the initial one only. As an example we shall discuss a curious partial generalization of a theorem belonging to the French mathematicians Victor Thébault (1882–1960), which is included in the selected 400 problems of the American journal "AMERICAN MATHEMATICAL MONTLY" (Alexeev, 1977). The assertion could be formulated in the following way:

Thébault's theorem. *If AH_a, BH_b and CH_c are the altitudes of a triangle ABC, then the Euler's lines of the triangles AH_bH_c, BH_cH_a and CH_aH_b have a common point T_0 on the nine-points circle for which one of the segments T_0H_a, T_0H_b and T_0H_c is equal to the sum of the other two* (Alexeev, 1977) (Fig. 1).

The point T_0 from the nine-points circle (Euler's circle) defined by the above Thébault's theorem could be named Thébault's point of the given triangle ABC. We will show a partial generalization of Thébault's theorem by central conics. Before this we will generalize two notions from the Geometry of the triangle.

We will obtain the proofs of all assertions by means of barycentric coordinates with respect to the given triangle ABC, namely $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$. Besides this the midpoints of the sides BC, CA and AB are denoted by $M_a\left(0,\frac{1}{2},\frac{1}{2}\right)$, $M_b\left(\frac{1}{2},0,\frac{1}{2}\right)$ and $M_c\left(\frac{1}{2},\frac{1}{2},0\right)$, respectively.

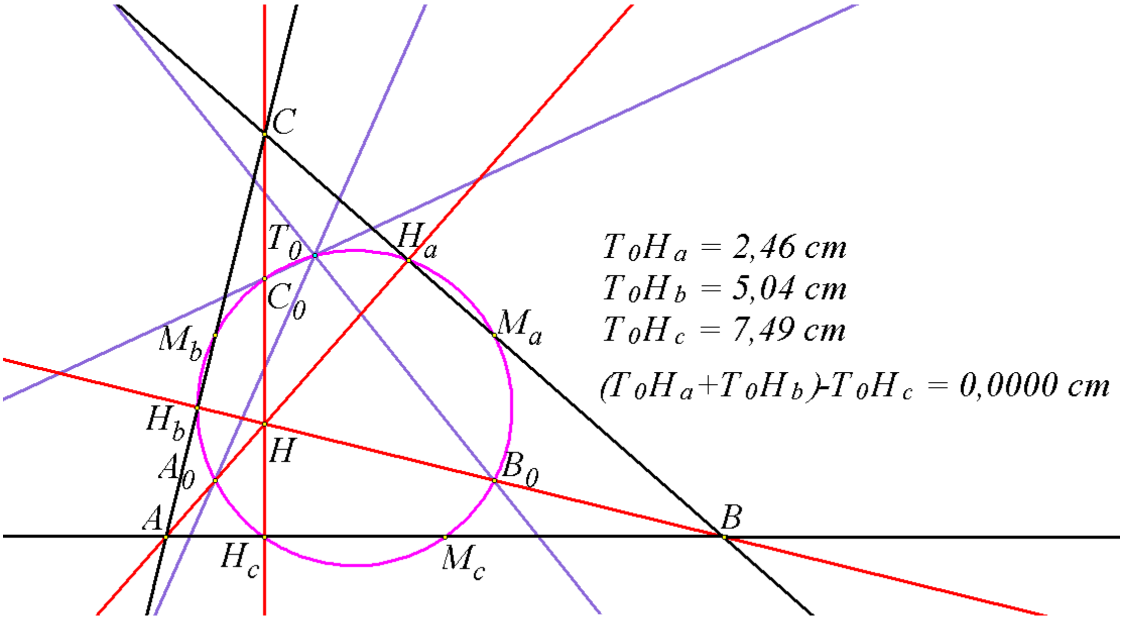

Figure 1

EULER'S LINE AND EULER'S CURVE DEPENDING ON A POINT

Every non-equilateral triangle has a special line which is called Euler's line and a special circle, which is called Euler's circle. In the general case the characteristic points of ΔABC, which are on the Euler's line, are the circum-center, the orthocenter, the center of gravity and the center of Euler's circle. The circumcircle of ΔABC is only an element of the infinity set of conics which are circumscribed with respect to ΔABC. Something more, if $O(x_0, y_0, z_0)$ $(x_0 + y_0 + z_0 = 1)$ is an arbitrary point which is not on any of the lines BC, CA and AB, M_bM_c, M_cM_a and M_aM_b in the plane of ΔABC, then A, B, C and their correspondingly symmetric points with respect to O: $A'(2x_0 - 1, 2y_0, 2z_0)$, $B'(2x_0, 2y_0 - 1, 2z_0)$ and $C'(2x_0, 2y_0, 2z_0 - 1)$ lie on a conic $\bar{k}(O)$ with center O (Fig. 2, 3), while the curve $\bar{k}(O)$ has the following equation

$$\bar{k}(O):(1-2x_0)x_0yz + (1-2y_0)y_0zx + (1-2z_0)z_0xy = 0.$$

The point O, determined in this way, is analogous to the circum-center of ΔABC. Now, we will determine a point which is analogous to the orthocenter of ΔABC. The altitudes of ΔABC are parallel to the lines connecting the circumcenter and the points M_a, M_b and M_c. This gives a ground to construct the lines h_a, h_b and h_c, passing through the vertices A, B and C, parallel to the lines OM_a, OM_b and OM_c, respectively. Note that they have a common point H (Fig. 2, 3). In addition, no matter how the location of O is changed, the lines under consideration have a common point. Thus, we obtain the following:

Property 1. *The lines* h_a, h_b and h_c *have a common point* H.

We leave the proof of this fact to the reader. Only note, that the point H has the following coordinate representation $H(1-2x_0, 1-2y_0, 1-2z_0)$.

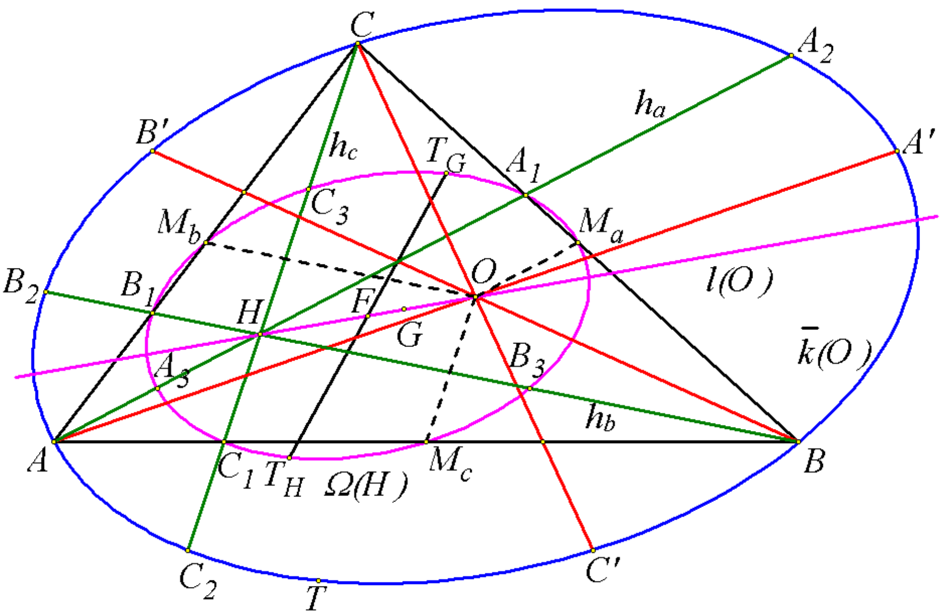

Figure 2

For the analogy between the point H and the orthocenter of ΔABC, other properties which are characteristic for the orthocenter should be checked for H. Let the lines h_a, h_b and h_c intersect the lines BC, CA and AB in the points A_1, B_1 and C_1, respectively, while they intersect the curve $\bar{k}(O)$ in A_2, B_2 and C_2, respectively (Fig. 2, 3). Then, we have

Property 2. *The points A_2 B_2 and C_2 are symmetric to H with respect to A_1, B_1 and C_1, correspondingly.*

The proofs of these properties could be obtained using the coordinate representations

$$A_1\left(0,\frac{1-2y_0}{2x_0},\frac{1-2z_0}{2x_0}\right),\ B_1\left(\frac{1-2x_0}{2y_0},0,\frac{1-2z_0}{2y_0}\right),\ C_1\left(\frac{1-2x_0}{2z_0},\frac{1-2y_0}{2z_0},0\right).$$

$$A_2\left(2x_0-1,\frac{(1-x_0)(1-2y_0)}{x_0},\frac{(1-x_0)(1-2z_0)}{x_0}\right),$$

$$B_2\left(\frac{(1-y_0)(1-2x_0)}{y_0},2y_0-1,\frac{(1-y_0)(1-2z_0)}{y_0}\right)$$

$$C_2\left(\frac{(1-z_0)(1-2x_0)}{z_0},\frac{(1-z_0)(1-2y_0)}{z_0},2z_0-1\right).$$

It is true the following

Property 3. *The points A', B' and C' are symmetric to H with respect to M_a, M_b and M_c, correspondingly.*

Because of the established analogy between the point H and the orthocenter of ΔABC, we will call H *orthoid* of ΔABC, *depending on the point O*.

If $G\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ is he center of gravity of ΔABC, note that $G\in OH$ and $GH:GO=2:1$. Thus, we have

Property 4. *The points O, H and G are colinear and the point G divides the segment HO in ratio* $2:1$.

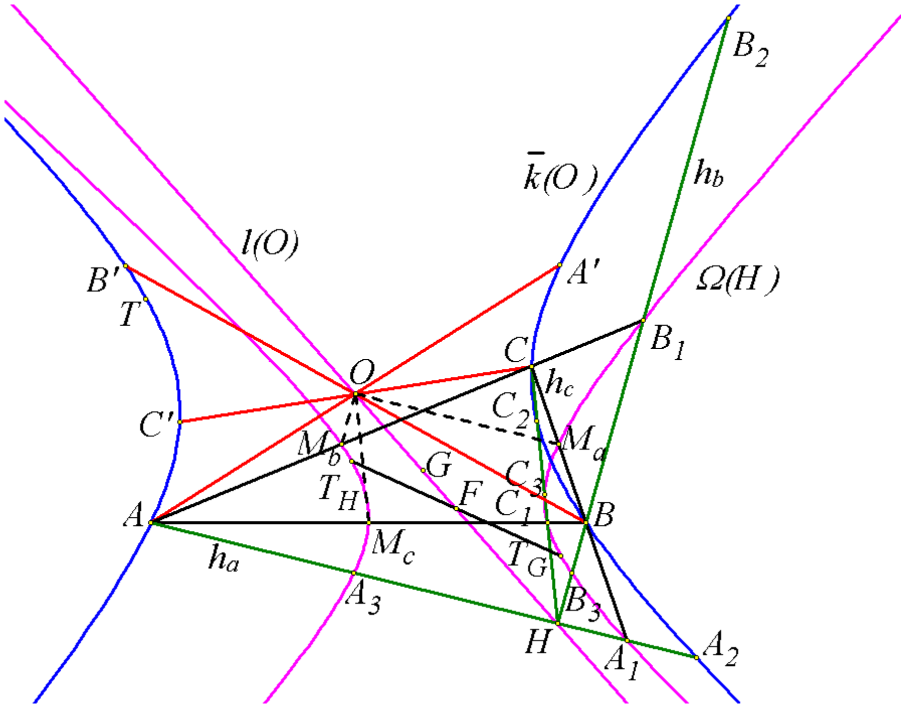

Figure 3

Property 4 shows that the line $l(O)$, passing through the points O, H and G, is an analogue of Euler's line and for this reason we will call $l(O)$ to be *Euler's line of* ΔABC, *depending on the point* O.

Euler's line contains the center of Euler's circle of ΔABC too and let us look for an analogous curve of Euler's circle. In the general case the points M_a, M_b and M_c lie on Euler's circle, the base points of the altitudes and the midpoints of the segments connecting the vertices and the orthocenter lie on it too.

For an arbitrary point $P(\lambda,\mu,\nu)$ $(\lambda+\mu+\nu=1)$ in the plane of ΔABC, the points M_a, M_b and M_c, the intersection points of the lines AP, BP and CP with the lines BC, CA and AB, respectively, and also the midpoints of the segments AP, BP and CP, denoted by A_3, B_3 and C_3, lie on a second degree curve $\Omega(P)$ (Fig. 2, 3), which is called *Euler's curve for the point* P *with respect to* ΔABC. This curve has the following equation:

$$\Omega(P): \mu\nu x^2 + \nu\lambda y^2 + \lambda\mu z^2 - (1-\lambda)\lambda yz - (1-\mu)\mu zx - (1-\nu)\nu xy = 0.$$

Obviously, Euler's circle of ΔABC is Euler's curve of its orthocenter. Since H is an analogue of the orthocenter, in this case it is appropriate to consider Euler's curve $\Omega(H)$ of the point H. It follows from the definition that $\Omega(P)$ passes through the points M_a, M_b, M_c, A_1, B_1, C_1 and the midpoints of the segments AP, BP, CP, while it follows from the last equality that its equation is

$$\Omega(H): \begin{array}{l}(1-2y_0)(1-2z_0)x^2 + (1-2z_0)(1-2x_0)y^2 + (1-2x_0)(1-2y_0)z^2 - \\ -2(1-2x_0)x_0 yz - 2(1-2y_0)y_0 zx - 2(1-2z_0)z_0 xy = 0.\end{array}$$

We call $\Omega(H)$ to be *Euler's curve of* ΔABC*, depending on the point* O.

The following is verified for the points A_3, B_3 and C_3

Property 5. *The lines* $A'A_3$, $B'B_3$ *and* $C'C_3$ *pass through the point* G *and the ratios* $GA' : GA_3 = GB' : GB_3 = GC' : GC_3 = 2:1$ *are satisfied.*

From the coordinates of the points under consideration we get the vector equalities $\overrightarrow{GA_3} = -\frac{1}{2}\overrightarrow{GA'}$, $\overrightarrow{GB_3} = -\frac{1}{2}\overrightarrow{GB'}$ и $\overrightarrow{GC_3} = -\frac{1}{2}\overrightarrow{GC'}$, which prove the property.

Let now $h\left(H, \frac{1}{2}\right)$ be a homothety with center H and coefficient $\frac{1}{2}$, while $h\left(G, -\frac{1}{2}\right)$ be a homothety with center O and coefficient $-\frac{1}{2}$. We obtain from properties 2, 3, 5 and the main property of the center of gravity that

Property 6. *The homotheties* $h\left(H, \frac{1}{2}\right)$ *and* $h\left(G, -\frac{1}{2}\right)$ *transform* $\bar{k}(O)$ *into* $\Omega(H)$.

It follows from the last property directly, that

Property 7. *The curves* $\bar{k}(O)$ *and* $\Omega(H)$ *are of one and the same kind.*

Let $F\left(\frac{1-x_0}{2}, \frac{1-y_0}{2}, \frac{1-z_0}{2}\right)$ be the midpoint of the segment OH. It follows from property 6, that

Property 8. *The point* F *is center of Euler's curve* $\Omega(H)$.

Let T be an arbitrary point on $\bar{k}(O)$. Construct the images T_H and T_G of this point with respect to $h\left(H, \frac{1}{2}\right)$ and $h\left(G, -\frac{1}{2}\right)$, respectively. Consider the line $T_H T_G$. Notice, that when moving the point T along $\bar{k}(O)$, the line $T_H T_G$ passes through F. Indeed, if S is an arbitrary point in the space, it follows from the definitions of the homotheties under consideration, that $\overrightarrow{ST_H} = \frac{1}{2}\left(\overrightarrow{ST} + \overrightarrow{SH}\right)$ and $\overrightarrow{ST_G} = \frac{1}{2}\left(-\overrightarrow{ST} + 3\overrightarrow{SG}\right)$. From here we obtain $\overrightarrow{FT_H} + \overrightarrow{FT_G} = \vec{o}$ when $S = F$. Thus, the following is proved

Property 9. *The homotheties* $h\left(H, \frac{1}{2}\right)$ *and* $h\left(G, -\frac{1}{2}\right)$ *transform an arbitrary point on* $\bar{k}(O)$ *to a diametrically opposite point on* $\Omega(H)$.(Fig. 2, 3)

It follows from the obtained results that the line $l(O)$ does not exist exactly when $O \equiv G$. In this case $\Omega(G)$ is an inscribed ellipse of ΔABC. This fact explains why the equilateral triangle has no Euler's line.

Until now we have considered the cases when the circumscribed curve of ΔABC is an ellipse or hyperbola with center in a given point O. The circumscribed parabolas of ΔABC could be considered as conics with infinity centers. The infinite center O of the parabola could be determined by the direction of a given vector $\vec{O}$. Let $\vec{O}(x_0, y_0, z_0)$ $(x_0 + y_0 + z_0 = 0)$ be a vector which is not colinear with any of the lines BC, CA and AB. A unique parabola $k(\vec{O})$ exists, which passes through the points A, B and C having a colinear axis with $\vec{O}$ (it is tangent to the infinite line of the plane at the infinite point O). The parabola $k(\vec{O})$ has the following equation

$$k(\vec{O}): x_0^2 yz + y_0^2 zx + z_0^2 xy = 0 .$$

In this case the point H could be considered coinciding with the infinite center of $k(\vec{O})$. The line $l(\vec{O})$ through G and colinear with $\vec{O}$ is considered as *Euler's line of* ΔABC*, depending on the direction of* $\vec{O}$ *(or depending on the infinite point* O*, which is the same). We denote it by* $l(O)$.

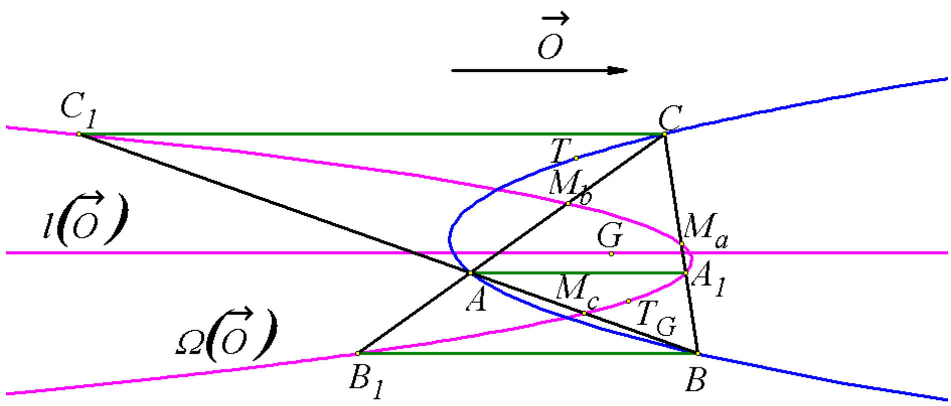

Figure 4

Construct the lines a_0, b_0 and c_0, passing through the vertices A, B and C, respectively, colinear with the vector $\vec{O}$. Let $a_0 \cap BC = A_1$, $b_0 \cap CA = B_1$ and $c_0 \cap AB = C_1$. The corresponding coordinates are $A_1\left(0, -\frac{y_0}{x_0}, -\frac{z_0}{x_0}\right)$, $B_1\left(-\frac{x_0}{y_0}, 0, -\frac{z_0}{y_0}\right)$, $C_1\left(-\frac{x_0}{z_0}, -\frac{y_0}{z_0}, 0\right)$. The coordinates of the points M_a, M_b, M_c, A_1, B_1, C_1 and the vector $\vec{O}$ satisfy the equation

$$\Omega(\vec{O}): y_0 z_0 x^2 + z_0 x_0 y^2 + x_0 y_0 z^2 + x_0^2 yz + y_0^2 zx + z_0^2 xy = 0 .$$

Consequently, they lie on the parabola $\Omega(\vec{O})$ whose axis is colinear with $\vec{O}$. The curve has the above equation (Fig. 4). We consider the parabola $\Omega(\vec{O})$ as *Euler's curve of* ΔABC*, depending on the direction of* $\vec{O}$ *(or depending on the infinity point, which is the same).*

If $T\left(x_T, y_T, z_T\right)$ is an arbitrary point on $k\left(\vec{O}\right)$, it could be noticed, that the point $T_G\left(\frac{1-x_T}{2}, \frac{1-y_T}{2}, \frac{1-z_T}{2}\right)$ lies on $\Omega\left(\vec{O}\right)$. Additionally, it is easy to check, that the equality $\overrightarrow{GT_G} = -\frac{1}{2}.\overrightarrow{GT}$ is satisfied. Therefore, it is true the following

Property 10. *The homothety* $h\left(G, -\frac{1}{2}\right)$ *transforms the parabola* $\Omega\left(\vec{O}\right)$ *to the parabola* $k\left(\vec{O}\right)$ (Fig. 4).

The last property shows that the circumscribed parabolas of ΔABC look like the circumscribed ellipse of ΔABC with center G taking into account that in both cases there is only one homothety transforming the circumscribed curve to its corresponding Euler's curve.

We see that the infinite points also generate Euler's lines and Euler's curves (parabolas) of a given triangle. However, these cases are less rich in properties since they are connected with circumscribed parabolas of the triangle. The parabolas occupy a more special place in the set of the three types conics with respect to the notion of a center. For this reason they give the corresponding peculiarity of the questions under consideration.

The above investigations show that every finite and every infinite point O from the plane of a given triangle ABC, which do not lie on any of the lines BC, CA, AB, B_0C_0, C_0A_0, A_0B_0 and which are different from the center of gravity G, could be connected with Euler's line $l\left(O\right)$ and Euler's curve $\Omega\left(H\right)$. The line $l\left(O\right)$ and the curve $\Omega\left(H\right)$ coincide with classic ones exactly when O is the center of the circumscribed circle of a non-equilateral triangle ABC.

Finally note, that if the considered constructions are done in a reverse order from the point H, we obtain the center O of a unique circumscribed conic of ΔABC. This means that both points are interchangeable in these constructions.

THÉBAULT'S POINT FOR EULER'S CURVE WHICH IS AN ELLIPSE OR A HYPERBOLA

Let P be an arbitrary point in the plane of ΔABC and $AP \cap BC = P_a$, $BP \cap CA = P_b$, $CP \cap AB = P_c$. Additionally, we denote the midpoints of the segments AP, BP and CP by A_0, B_0 and C_0, respectively. The triangles AP_bP_c, BP_cP_a and CP_aP_b correspond to the triangles AH_bH_c, BH_cH_a and CH_aH_b in Thébault's theorem when changing the orthocenter H by the point P. Changing H by P, we change Euler's circle by Euler's curve $\Omega(P)$, generated by the point P. The points A_0, B_0 and C_0 are centers of unique circumscribed conics of the triangles AP_bP_c, BP_cP_a and CP_aP_b, which are the centers of the corresponding circumcircles of the triangles AH_bH_c, BH_cH_a and CH_aH_b when $P \equiv H$. If G_a, G_b and G_c are the centers of gravity of the triangles AP_bP_c, BP_cP_a and CP_aP_b, respectively, then the lines $l_a = A_0G_a$, $l_b = B_0G_b$ and C_0G_c are their Euler's lines determined by the point P. For this reason we could expect that these lines have a common point on the curve $\Omega(P)$. More precisely, the expected result is formulated in the following way:

Theorem. *If P is a finite point in the plane of ΔABC, which is different from its center of gravity and does not lie on any of the lines BC, CA and AB, then the lines A_0G_a, B_0G_b and C_0G_c have a common point T on Euler's curve $\Omega(P)$ or they are parallel* (Fig. 1, 5).

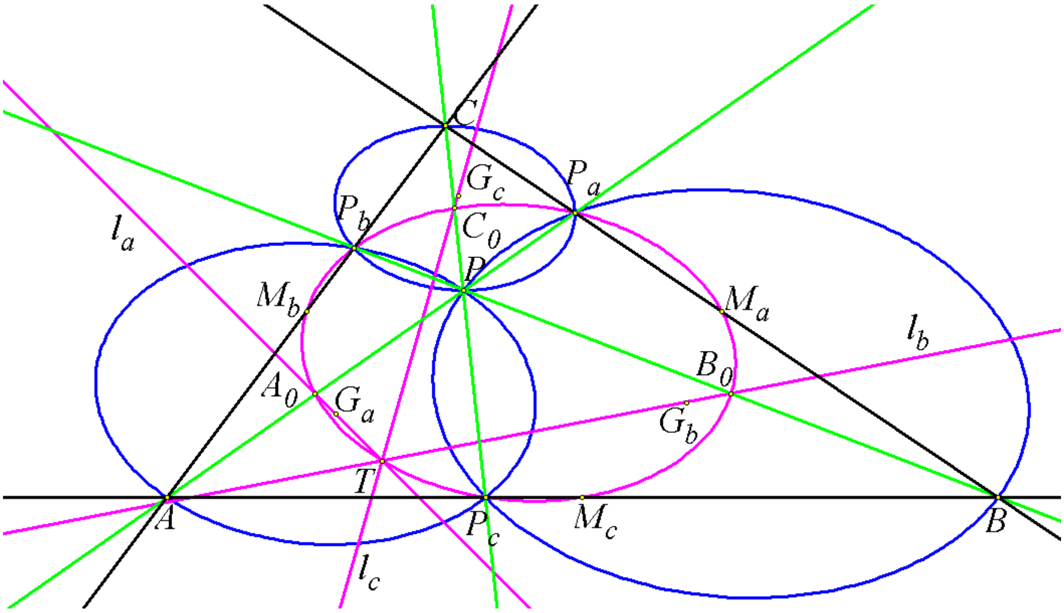

Figure 5

Firstly note that when P is the center of gravity of ΔABC, the lines under consideration have a common point but this point does not lie on $\Omega(P)$ (this is the ellipse which is tangent to the sides of ΔABC at the points M_a, M_b and M_c) and coincides with its center P. For this reason the center of gravity of ΔABC is excluded from the considerations.

Let the coordinate representation of the point P be the following $P(\lambda,\mu,\nu)$ $(\lambda+\mu+\nu=1)$. Then

(1) $$P_a\left(0,\frac{\mu}{\mu+\nu},\frac{\nu}{\mu+\nu}\right),\ P_b\left(\frac{\lambda}{\nu+\lambda},0,\frac{\nu}{\nu+\lambda}\right),\ P_c\left(\frac{\lambda}{\lambda+\mu},\frac{\mu}{\lambda+\mu},0\right).$$

(2) $$A_0\left(\frac{\lambda+1}{2},\frac{\mu}{2},\frac{\nu}{2}\right),\ B_0\left(\frac{\lambda}{2},\frac{\mu+1}{2},\frac{\nu}{2}\right),\ C_0\left(\frac{\lambda}{2},\frac{\mu}{2},\frac{\nu+1}{2}\right)$$

The equation of $\Omega(P)$ is presented in the following way

(3) $$\mu\nu x^2+\nu\lambda y^2+\lambda\mu z^2-(1-\lambda)\lambda yz-(1-\mu)\mu zx-(1-\nu)\nu xy=0.$$

From (1) we obtain for the coordinates of the centers of gravity G_a, G_b and G_c that:

(4) $$G_a\left(\frac{\lambda^2+2\lambda+\mu\nu}{3(1-\mu)(1-\nu)},\frac{\mu}{3(1-\nu)},\frac{\nu}{3(1-\mu)}\right),G_b\left(\frac{\lambda}{3(1-\nu)},\frac{\mu^2+2\mu+\nu\lambda}{3(1-\mu)(1-\nu)},\frac{\nu}{3(1-\lambda)}\right),$$
$$G_c\left(\frac{\lambda}{3(1-\mu)},\frac{\mu}{3(1-\lambda)},\frac{\nu^2+2\nu+\lambda\mu}{3(1-\lambda)(1-\mu)}\right).$$

Now from (2) and (4) the parametric equations of the lines A_0G_a, B_0G_b and C_0G_c become:

(5) $$A_0G_a: \quad x=\frac{\lambda+1}{2}+\frac{\lambda^2-\lambda+\mu\nu+3\lambda\mu\nu}{(1-\mu)(1-\nu)}l_1,\ y=\frac{\mu}{2}+\frac{(1-3\nu)\mu}{1-\nu}l_1,\ z=\frac{\nu}{2}+\frac{(1-3\mu)\nu}{1-\mu}l_1,$$

(6) $$B_0G_b: \quad x=\frac{\lambda}{2}+\frac{(1-3\nu)\lambda}{1-\nu}l_2,\ y=\frac{\mu+1}{2}+\frac{\mu^2-\mu+\nu\lambda+3\lambda\mu\nu}{(1-\nu)(1-\lambda)}l_2,\ z=\frac{\nu}{2}+\frac{(1-3\lambda)\nu}{1-\lambda}l_2,$$

(7) $$C_0G_c: \quad x=\frac{\lambda}{2}+\frac{(1-3\mu)\lambda}{1-\mu}l_3,\ y=\frac{\mu}{2}+\frac{(1-3\lambda)\mu}{1-\lambda}l_3,\ z=\frac{\nu+1}{2}+\frac{\nu^2-\nu+\lambda\mu+3\lambda\mu\nu}{1-\lambda}l_3.$$

Let the lines A_0G_a and B_0G_b intersect in the point T. Then, after solving the system $(5)-(6)$ we get that their common point T has the following coordinates:

(8) $$x_T=\frac{\lambda(\mu-\nu)^2}{\tau},\ y_T=\frac{\mu(\nu-\lambda)^2}{\tau},\ y_T=\frac{\nu(\lambda-\mu)^2}{\tau},$$

where $\tau=\lambda\mu+\mu\nu+\nu\lambda-9\lambda\mu\nu$.

After replacing (8) in (7), it is easy to establish that T lies on the line C_0G_a too. Additionally, replacing the coordinates (8) in (3) we conclude that $T\in\Omega(P)$.

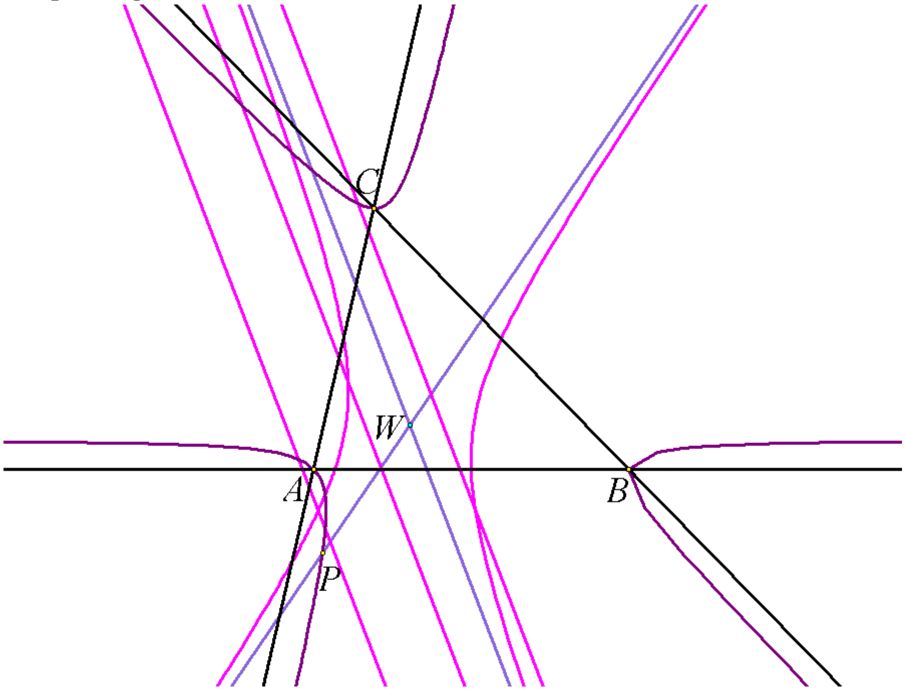

Figure 6

If the lines A_0G_a and B_0G_b are parallel, then the coefficients in front of l_1 and l_2 are proportional. We obtain that $\tau=0$. Assuming that the lines A_0G_a and C_0G_c intersect, then the equalities (8) are satisfied for their common point as it was established already. The equalities are possible only in the case $\tau\neq 0$. Consequently, the lines A_0G_a, B_0G_b and C_0G_c are parallel (Fig. 6). Thus, the theorem is proved completely. It was shown in the proof that it was true the following:

Corollary 1. *The lines* A_0G_a, B_0G_b *and* C_0G_c *are parallel exactly when the point* P *lies on the third order curve with equation*

(9) $$K:\lambda\mu+\mu\nu+\nu\lambda-9\lambda\mu\nu=0.$$

Since $\lambda,\mu,\nu\in\left(-\infty,\frac{1}{9}\right)\cup\left[1,+\infty\right)$, it follows from the geometric meaning of the coordinates λ, μ, ν that in the case $P\in K$ Euler's curve $\Omega(P)$ is a hyperbola.

Corollary 2. *If the point* P *is on* K*, the lines* A_0G_a, B_0G_b *and* C_0G_c *are parallel to one of the asymptotes, while* P *lies on the other asymptote of the hyperbola* $\Omega(P)$ (Fig. 6).

The first part of corollary 2 should be expected to be true since it is definitely in accordance with the intersection of the lines A_0G_a, B_0G_b and C_0G_c on $\Omega(P)$. For this reason the first part of corollary 2 could be treated in the following way: the common infinite point of the lines A_0G_a, B_0G_b and C_0G_c lies on $\Omega(P)$.

The infinite points of $\Omega(P)$ are determined by the general solutions of the system defined by $x+y+z=0$ and (3). The following homogeneous equation is obtained

(10) $$\nu(1-\nu)y^2+2\mu\nu yz+\mu(1-\mu)z^2=0.$$

In order to prove that the line A_0G_a is parallel to one of the asymptotes of $\Omega(P)$, we have to prove that the coordinates of a colinear vector with A_0G_a satisfy (10). From (5) (the coefficients in front of l_1) and (9) (the point P is on K) we obtain that the vector $\left(2\mu\nu(3\lambda-1),\mu(3\nu-1)(1-\mu),\nu(3\mu-1)(1-\nu)\right)$ is colinear with A_0G_a. Now, substitute the coordinates of this vector into the left hand side of (10) and take into account that λ, μ, ν satisfy (9) to obtain the true equality.

The center of $\Omega(P)$ is the point $W\left(\frac{\lambda+1}{4},\frac{\mu+1}{4},\frac{\nu+1}{4}\right)$ (Fig. 6). In order to prove that P is a point on the asymptote of $\Omega(P)$, we have to show that the coordinates of a colinear vector with WP satisfy (10). A colinear vector with WP is $(3\lambda-1,3\mu-1,3\nu-1)$. As in the previous case it is easy to check that this vector satisfies (10).

It remains to show that the point P does not lie on the same asymptote which is parallel to the line A_0G_a. Assuming the contrary we conclude that the vectors considered previously are colinear, i.e.

$$\frac{2\mu\nu(3\lambda-1)}{3\lambda-1}=\frac{\mu(3\nu-1)(1-\mu)}{3\mu-1}=\frac{\nu(3\mu-1)(1-\nu)}{3\nu-1}.$$

The only solution of these equalities is $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, which presents the center of gravity of ΔABC, excluded from considerations. Thus, corollary 2 is proved completely.

The proved theorem and the complementary corollaries ground calling the common point (finite or infinite) of the lines A_0G_a, B_0G_b and C_0G_c *to be Thébault's point for Euler's curve* $\Omega(P)$. It is seen from the corollaries that *Thébault's point is infinite exactly when* P *lies on the curve* K.

CONCLUSION

Note that the original Thébault's theorem contains three assertions:

1) Euler's lines of the triangles AH_bH_c, BH_cH_a and CH_aH_b have a common point;

2) The common point T_0 of these lines lies on Euler's circle of ΔABC;

3) The sum of two of distances T_0H_a, T_0H_b and T_0H_c is equal to the third one.

A generalization of assertion 1) only, with circles and isogonally conjugated points is done in (Nenkov, 2020).

By the proved theorem we have shown how to generalize assertions 1) and 2) simultaneously.

The generalization of assertion 3) is the most difficult. There exist points in the plane of the triangle for which assertion 3) is satisfied. In the general case it is very difficult to define points P, for which one of the distances TP_a, TP_b and TP_c is equal to the sum of the other two. Fig. 7 shows a case in which the tringle is equilateral and the assertion 3) is satisfied.

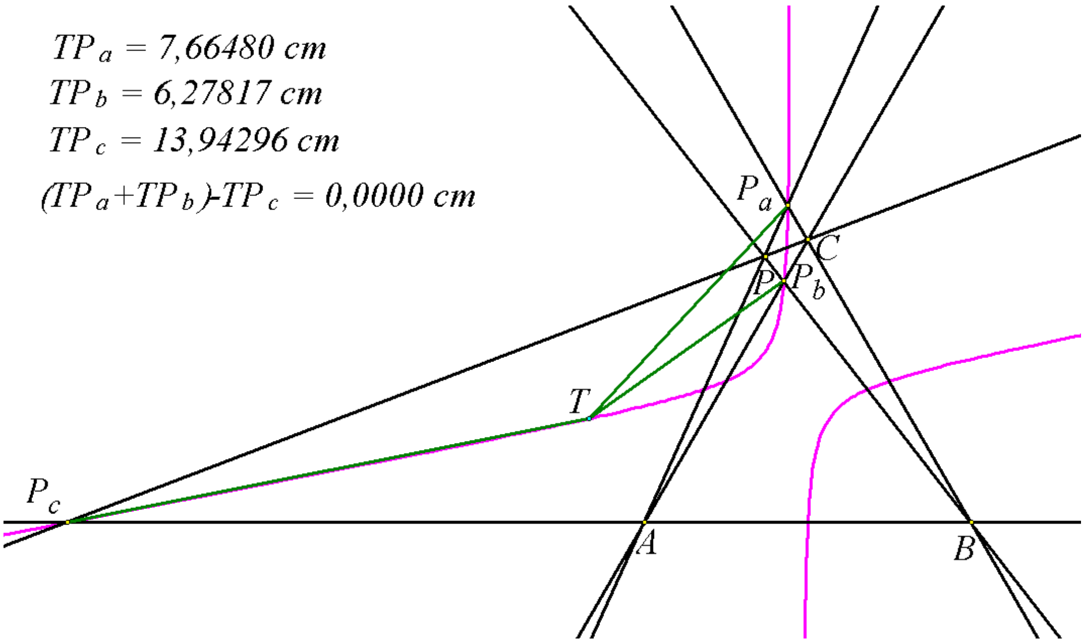

Figure 7

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