International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 6, 2021, pp. 43–67
Received 15 February 2021. Published on-line 15 April 2021
web: http://www.journal-1.eu/

Equilateral Triangles formed by the Centers of Erected Triangles

Stanley Rabinowitz[a] and Ercole Suppa[b]
[a] 545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net
web: http://www.StanleyRabinowitz.com/
[b] Via B. Croce 54, 64100 Teramo, Italia
e-mail: ercolesuppa@gmail.com

Abstract. Related triangles are erected outward on the sides of a triangle. A triangle center is constructed in each of these triangles. We use a computer to find instances where these three centers form an equilateral triangle.

Keywords. triangle geometry, erected triangles, equilateral triangles, computer-discovered mathematics, GeometricExplorer.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

The well-known result, known as Napoleon's Theorem, states that if equilateral triangles are erected outward on the sides of an arbitrary triangle ABC, then their centers (G, H, and I) form an equilateral triangle (Figure 1).

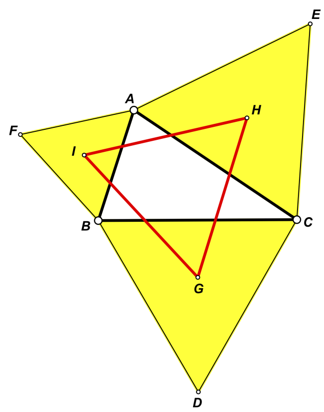


Figure 1. Outer Napoleon Triangle

Triangle GHI is called the *outer Napoleon triangle* of $\triangle ABC$. As is also well known, the equilateral triangles can be erected inward (Figure 2). In this case, the equilateral triangle formed is called the *inner Napoleon triangle* of $\triangle ABC$.

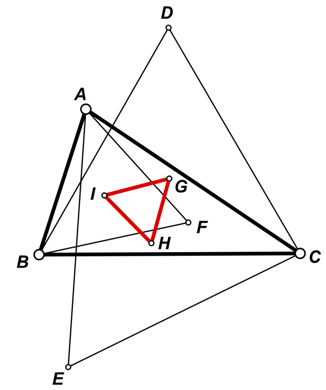


FIGURE 2. Inner Napoleon Triangle

In this paper, we look for generalizations of Napoleon's Theorem. We will erect triangles on the sides of an arbitrary triangle and then pick triangle centers associated with these triangles. We look for instances where these three triangle centers form an equilateral triangle.

We used a computer program, GeometricExplorer, to vary the type of triangle center used from X_1 to X_{1000}, omitting points at infinity. Then the program checked to see if the three centers formed an equilateral triangle.

In this paper, we report on the results discovered by our computer program. If the result has previously appeared in the literature, we give a reference. In most cases, the results can be proven analytically using barycentric coordinates. However, the calculations frequently involve very complicated algebraic expressions, so we omit the details. However, if a simple geometric proof is known, we will present it.

2. ISOSCELES TRIANGLES

Instead of erecting equilateral triangles on the sides of $\triangle ABC$ as in Napoleon's Theorem, we can erect isosceles triangles. The triangles are erected so that their bases are the sides of $\triangle ABC$. Specifically, $\angle CBD = \angle BCD$, $\angle ACE = \angle CAE$, and $\angle BAF = \angle ABF$ as shown in Figure 3. The triangles may be erected outward or inward.

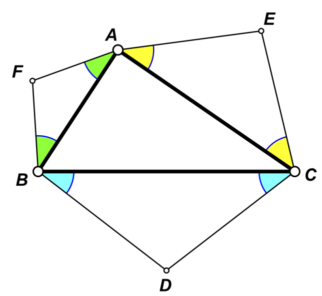


FIGURE 3. Isosceles Triangles

Our program found the following two results.

Theorem 2.1. *Isosceles triangles BCD, CAE, and ABF are erected on the sides of an arbitrary triangle ABC with their bases being the sides of $\triangle ABC$. Points G, H, and I are the X_{13} points of these triangles (Figure 4). Then $\triangle GHI$ is an equilateral triangle.*

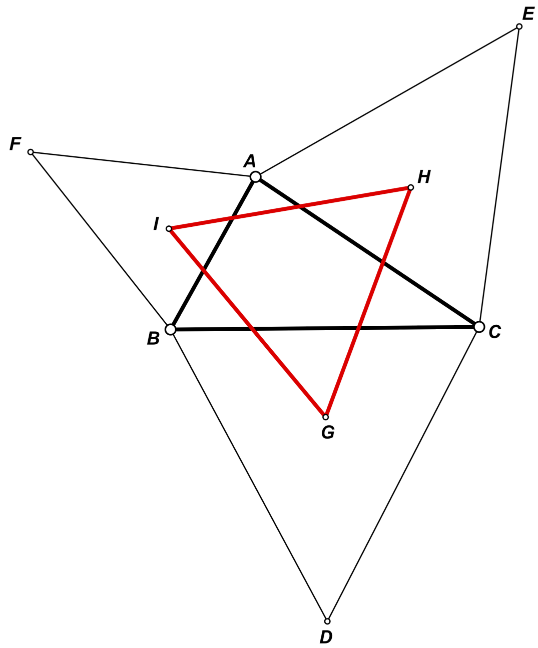
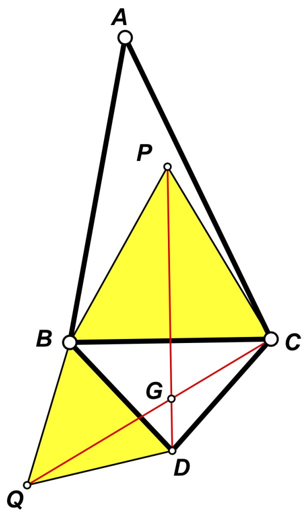


Figure 4. X_{13} centers of isosceles triangles

Proof. By definition, center X_{13}, also known as the first Fermat point, is the point of concurrence of lines from the vertices of a triangle to the outer vertices of equilateral triangles erected outward on the sides of the triangle.



Let BCD be an isosceles triangles erected outward on side BC of an arbitrary triangle ABC. Erect equilateral triangles BCP and BDQ outward on sides BC and BD of $\triangle BCD$. Lines PD and CG meet at G, the X_{13} center of $\triangle BCD$. Triangles PBD and CBQ are congruent because $PB = CB$, $BD = BQ$, and $\angle PBD = \angle CBD + 60° = \angle CBQ$. Since these triangle are congruent, $\angle BPD = \angle BCQ$, so $\angle BCG = 30°$. It therefore follows that $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$. $\square$

Theorem 2.2. *Isosceles triangles BCD, CAE, and ABF are erected on the sides of an arbitrary triangle ABC with their bases being the sides of $\triangle ABC$. Points G, H, and I are the X_{14} points of these triangles (Figure 5). Then $\triangle GHI$ is an equilateral triangle.*

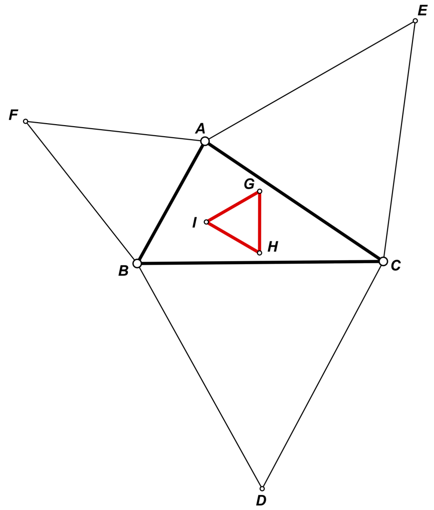


FIGURE 5. X_{14} centers of isosceles triangles

Proof. The proof is similar. Again we find that $\angle BCG = 30°$ and $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$. □

We can rephrase Napoleon's Theorem in terms of isosceles triangles.

Theorem 2.3. *Isosceles triangles BCG, CAH, and ABI are erected outward on the sides of $\triangle ABC$ with their bases being the sides of $\triangle ABC$. If the base angles of the isosceles triangles are all $30°$, then $\triangle GHI$ is an equilateral triangle (Fig. 6).*

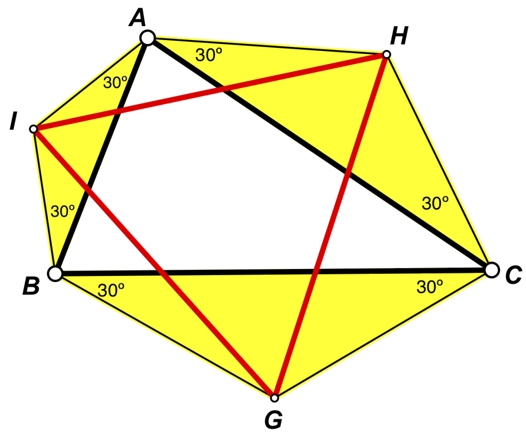


FIGURE 6. An equivalent version of Napoleon's Theorem

With this formulation, we can state a generalization, discovered by computer.

Theorem 2.4. *Isosceles triangles BCG, CAH, and ABI are erected outward on the sides of an arbitrary triangle ABC with their bases being the sides of $\triangle ABC$. Then $\triangle GHI$ is an equilateral triangle if and only if $HI \perp AX_{13}$, $GI \perp BX_{13}$, and $GH \perp CX_{13}$, where X_{13} is the 1st Fermat point of $\triangle ABC$ (Figure 7).*

This result can be proven geometrically, by noting that the lines from the vertices of a triangle through the X_{13} point form angles of $60°$ with each other.

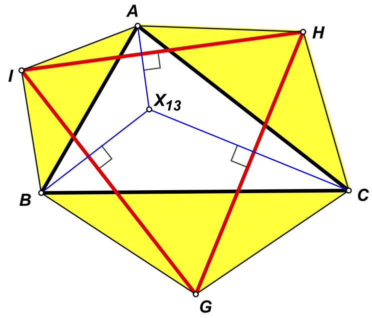


FIGURE 7. Isosceles triangles erected outward

If the isosceles triangles are erected inward, then a similar result holds with X_{13} replaced by X_{14}.

Theorem 2.5. *Isosceles triangles BCG, CAH, and ABI are erected inward on the sides of an arbitrary triangle ABC with their bases being the sides of $\triangle ABC$. Then $\triangle GHI$ is an equilateral triangle if and only if $HI \perp AX_{14}$, $GI \perp BX_{14}$, and $GH \perp CX_{14}$, where X_{14} is the 2nd Fermat point of $\triangle ABC$ (Figure 8).*

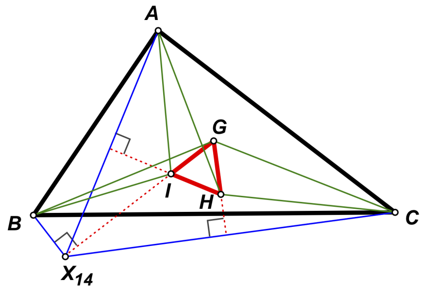


FIGURE 8. Isosceles triangles erected inward

3. SIMILAR TRIANGLES

Another way to generalize Napoleon's Theorem is to replace the equilateral triangles with similar triangles. We can construct similar triangles BCD, CAE, and ABF by taking $\angle CBD = \angle ACE = \angle BAF$ and $\angle BCD = \angle CAE = \angle ABF$ as shown in Figure 9. The triangles may be erected outward or inward.

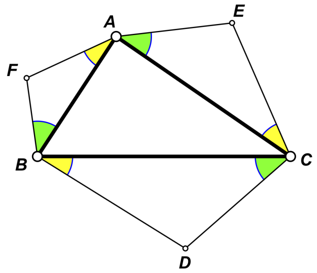


FIGURE 9. Similar Triangles

Our program did not find any results other than Napoleon's Theorem.

Theorem 3.1. *Let ABC be a fixed nonequilateral triangle. Similar triangles BCD, CAE, and ABF are erected (internally or externally) on the sides of $\triangle ABC$. Corresponding points G, H, and I are chosen in these triangles. Then $\triangle GHI$ is equilateral if and only if triangles BCG, CAH, and ABI are isosceles triangles with base on the sides of $\triangle ABC$ and base angles of $30°$. In that case $\triangle GHI$ is the inner or outer Napoleon triangle of $\triangle ABC$.*

Proof. Since G, H, and I are corresponding points of the similarity, triangles BCG, CAH, and ABI are similar. The result then follows from [14]. See also [5] and Corollary 2.1 of [16]. □

Another way of constructing similar triangles on the sides of $\triangle ABC$ is by taking $\angle CBD = \angle ABF = \angle CEA$, $\angle ACE = \angle BCD = \angle AFB$, and $\angle BAF = \angle EAC = \angle BDC$ as shown in Figure 10. The triangles may be erected outward or inward. We call these *similar Jacobi triangles.*

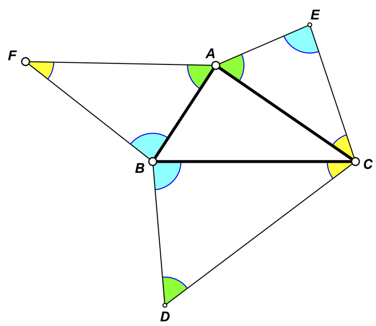


FIGURE 10. Similar Jacobi Triangles

Our program found the following result.

Theorem 3.2. *Similar Jacobi triangles BCD, CAE, and ABF are erected (internally or externally) on the sides of $\triangle ABC$. Points G, H, and I are the X_{15} points of these triangles (Figure 11). Then $\triangle GHI$ is equilateral. This triangle will be referred to as an outer Jacobi equilateral triangle.*

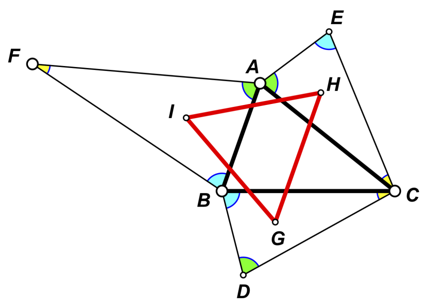


FIGURE 11. X_{15} centers of Similar Jacobi Triangles

In Napoleon's Theorem, for a given triangle ABC, two equilateral triangles are formed, the outer Napoleon triangle, N_o, and the inner Napoleon triangle, N_i. It is known how these triangles are related to $\triangle ABC$. The center of both triangles is the centroid of $\triangle ABC$. Triangles N_o and ABC are in perspective with perspector $X_{17}[ABC]$. Triangles N_i and ABC are in perspective with perspector $X_{18}[ABC]$.

Open Question 1. *How is the outer Jacobi equilateral triangle related to $\triangle ABC$?*

A partial result that our program found is that $\triangle ABD \sim \triangle IBG$.

Our program also discovered the following related result.

Theorem 3.3. *Similar triangles BCD, CAE, and ABF are erected (internally or externally) on the sides of $\triangle ABC$ as shown in Figure 12.*

Points G, H, and I are the X_{16} points of these triangles. Then $\triangle GHI$ is equilateral. This triangle will be referred to as an inner Jacobi equilateral triangle.

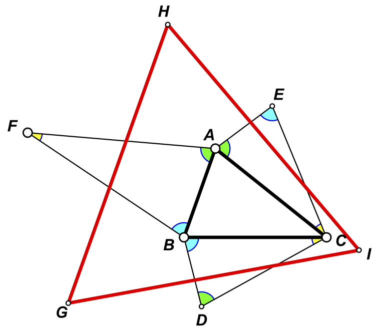


Figure 12. X_{16} centers of Similar Jacobi Triangles

Open Question 2. *How is the inner Jacobi equilateral triangle related to $\triangle ABC$?*

It is known that the areas of the inner and outer Napoleon triangles are connected by the formula $[N_o] - [N_i] = [\triangle ABC]$.

Open Question 3. *How are the inner and outer Jacobi equilateral triangles related?*

A special case of similar Jacobi triangles occurs when the three triangles are all similar to $\triangle ABC$. Specifically, $\angle CBD = \angle ABC = \angle ABF = \angle CEA$, $\angle ACE = \angle ACB = \angle BCD = \angle AFB$, and $\angle BAF = \angle BAC = \angle EAC = \angle BDC$ as shown in Figure 13. Observe that D is the reflection of A about BC, E is the reflection of B about CA, and F is the reflection of C about AB.

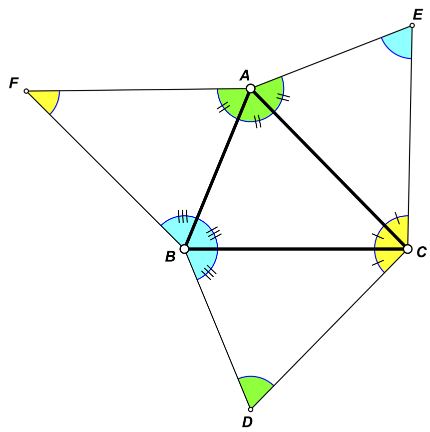


Figure 13.

In this case, we have the following result.

Theorem 3.4. *Triangles BCD, CAE, and ABF are erected externally on the sides of $\triangle ABC$ and are all similar to $\triangle ABC$ as shown in Figure 13. If points G, H, and I are the X_{15} points of these triangles, then $\triangle GHI$ is equilateral with center $X_{13}[ABC]$. Triangle GHI is homothetic to the outer Napoleon triangle of $\triangle ABC$ with $X_{16}[ABC]$ being the center of the homothety. If points G, H, and I are the X_{16} points of these triangles, then $\triangle GHI$ is equilateral with center $X_{14}[ABC]$. In this case, $\triangle GHI$ is homothetic to the inner Napoleon triangle of $\triangle ABC$ with $X_{15}[ABC]$ being the center of the homothety.*

4. Equal Outer Angles

Another way to generalize Napoleon's Theorem is to erect triangles on the sides of $\triangle ABC$ with only the restriction that the outer angles have measure $60°$. By the outer angles, we mean angles at D, E, and F as shown in Figure 14. The triangles need not be similar. Our program found the following results.

Theorem 4.1. *Triangles BCD, CAE, and ABF are erected externally on the sides of $\triangle ABC$ so that $\angle BDC = \angle CEA = \angle AFB = 60°$ as shown in Figure 14. Points G, H, and I are the circumcenters (X_3 points) of these triangles. Then $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$. If the three triangles are erected inward, then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$.*

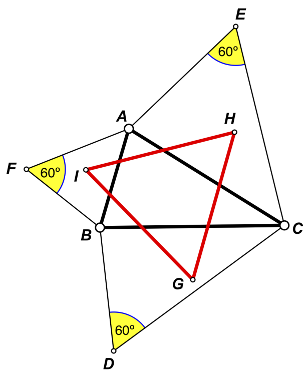


FIGURE 14. Circumcenters

Theorem 4.1 (with triangles erected outward) follows immediately from the following lemma. The proof is similar for the case when the triangles are inward.

Lemma 4.2. *Let BCD be a triangle with $\angle BDC = 60°$. Then the circumcenter of $\triangle BDC$ is the center of the equilateral triangle erected inward on side BC.*

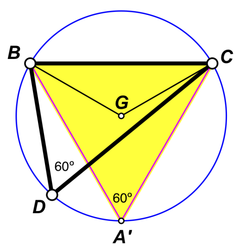


FIGURE 15.

Proof. Let BCA' be the equilateral triangle erected inward on side BC of $\triangle DBC$ (Figure 15). Since the angles at D and A' are equal, point A' lies on the circumcircle of $\triangle BDC$. Thus, triangles BCD and BCA' share the same circumcenter G. Therefore G is the center of equilateral triangle BCA'. □

Theorem 4.3. *Triangles BCD, CAE, and ABF are erected externally on the sides of $\triangle ABC$ so that $\angle BDC = \angle CEA = \angle AFB = 60°$ as shown in Figure 16. Points G, H, and I are the X_{110} points of these triangles. Then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$. If the three triangles are erected inward, then $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$.*

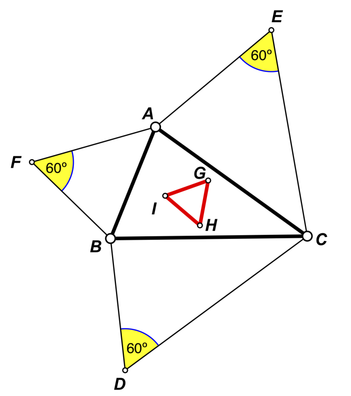


Figure 16. X_{110} points

Theorem 4.3 (with triangles erected outward) follows immediately from the following lemma. The proof is similar for the case when the triangles are erected inward.

Lemma 4.4. *Let BDC be a triangle with $\angle BDC = 60°$ (Figure 17). Then the X_{110} point of $\triangle BDC$ is the center of the equilateral triangle erected outward on side BC.*

Proof. Let O be the circumcenter of $\triangle BCD$. Let B', C', and D' be the reflections of O about the sides of $\triangle BCD$ (Figure 17).

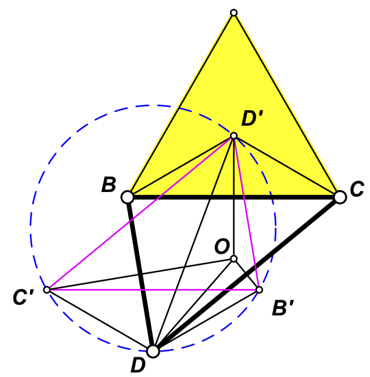


Figure 17.

Triangle $B'C'D'$ is the image of the medial triangle of BCD under a homothety with center O and ratio 2, so $\triangle BCD \cong \triangle B'C'D'$. Thus, $\angle C'D'B' = 60°$.

Since DC is the perpendicular bisector of OB' and DB is the perpendicular bisector of OC', this means $\angle B'DO = 2\angle CDO$ and $\angle C'DO = 2\angle BDO$. Thus, $\angle C'DB' = 2\angle BDC = 120°$.

Since $\angle C'D'B' + \angle C'DB' = 180°$, this means that points C', D, B', and D' are concyclic or that D' lies on $\odot C'DB'$.

Thus, $\odot C'DB'$ meets circles $\odot B'CD'$ and $\odot C'BD'$ at D'. But according to property (1) of [8], the circumcircles of triangles $C'D'B'$, $B'CD'$, and $C'BD'$ meet at the X_{110} point of $\triangle BCD$. Consequently, D' is the X_{110} point of $\triangle BCD$. Since $\angle D'BC = \angle D'CB = 30°$, D' is the center of the equilateral triangle erected outward on side BC. □

Note: When one angle of a triangle has measure $60°$, the X_{953} point of that triangle coincides with the X_{110} point. See [15].

Instead of using $60°$ for the measure of the outer angles, we can try other values. Our program found the following results, where the outer vertices have measure $30°$. Specifically, $\angle BDC = \angle CEA = \angle AFB = 30°$. The triangles may be erected outward or inward.

Theorem 4.5. *Triangles BCD, CAE, and ABF are erected externally on the sides of $\triangle ABC$ so that $\angle BDC = \angle CEA = \angle AFB = 30°$ as shown in Figure 18. If G, H, and I are the Kosnita points (X_{54} points) of these triangles, then $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$. If G, H, and I are the X_{195} points of these triangles, then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$. If the three triangles are erected inward and G, H, and I are the X_{54} points of these triangles, then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$. If the three triangles are erected inward and G, H, and I are the X_{195} points of these triangles, then $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$.*

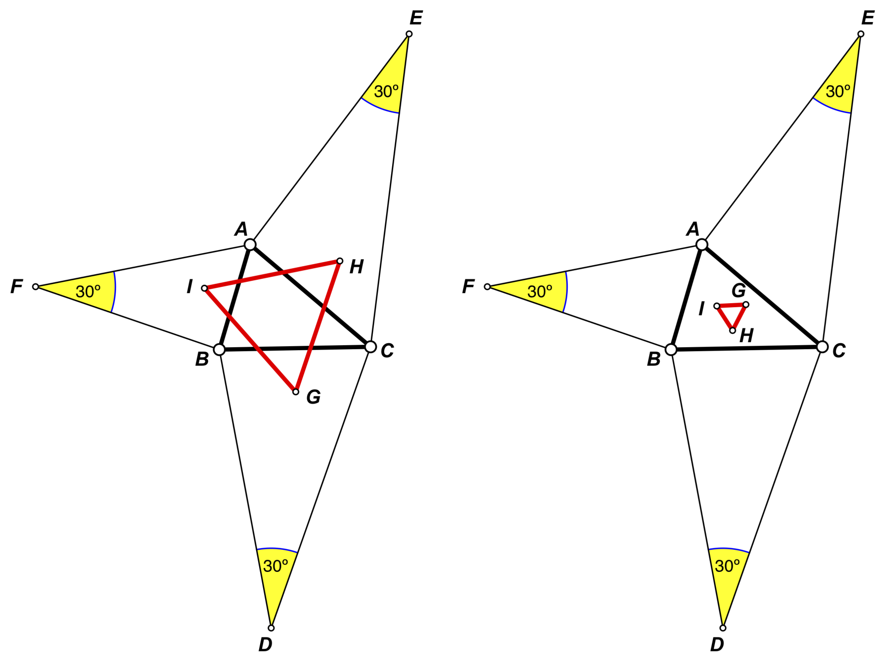
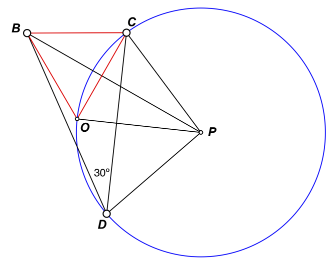


Figure 18. X_{54} Points (left); X_{195} Points (right)

Before we prove Theorem 4.5, we state a lemma.

Lemma 4.6. *In $\triangle BCD$, $\angle BDC = 30°$. Let O be the circumcenter of $\triangle BCD$ and let P be the circumcenter of $\triangle COD$. Then $\angle CBP = 30°$.*



Proof. Since $\angle BDC = 30°$, we have $\angle BOC = 60°$. Since $OB = OC$, $\triangle BCO$ is equilateral and $\angle CBO = 60°$. Segment CO is the common chord of $\odot CBO$ and $\odot COD$. Thus, BP is the perpendicular bisector of CO and BP bisects $\angle CBO$ which implies $\angle CBP = 30°$. □

We can now prove Theorem 4.5 for the cases where the triangles are erected outward. The proof is similar for the case when the triangles are erected inward.

Proof. Let G be the X_{54} point of $\triangle BCD$ where $\angle BDC = 30°$. It suffices to prove that $\angle CBG = 30° = \angle BCG = 30°$. Let O be the circumcenter of $\triangle BCD$ and let P be the circumcenter of $\triangle COD$. (Figure 19). According to [7], the line BP passes through the Kosnita point (X_{54}) of $\triangle BCD$. Thus $\angle CBG = 30°$ by the lemma. Similarly, $\angle BCG = 30°$ and we have shown that when G is the X_{54} point of $\triangle BCD$, then G is the outer Napoleon point of $\triangle ABC$.

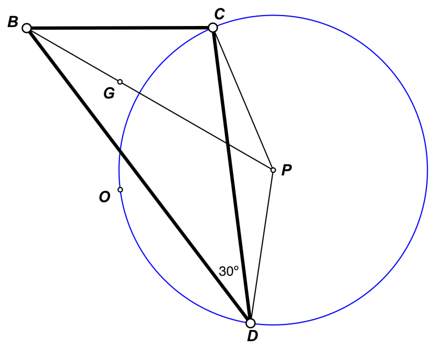


Figure 19.

Now suppose that G' is the X_{195} point of $\triangle BCD$. According to [9], the X_{195} point of a triangle is the reflection of the X_{54} point about the circumcenter of the triangle. Since G lies on the perpendicular bisector of BC, this implies that G' also lies on this perpendicular bisector and that G' is the inner Napoleon point of $\triangle ABC$. □

Our program also discovered similar results when the outer angle is 120°.

Theorem 4.7. *Triangles BCD, CAE, and ABF are erected externally on the sides of $\triangle ABC$ so that $\angle BDC = \angle CEA = \angle AFB = 120°$ as shown in Figure 20. If G, H, and I are the circumcenters (X_3 points) of these triangles, then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$. If the three triangles are erected inward, then $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$.*

Theorem 4.7 (with the triangles erected outward) follows immediately from the following lemma. The proof is similar for the case when the triangles are erected inward.

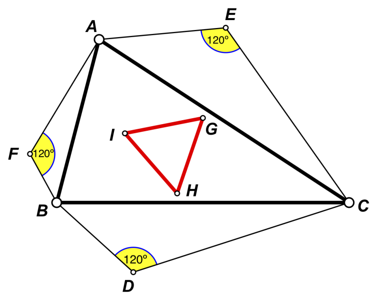


FIGURE 20. Circumcenters

Lemma 4.8. *Let BDC be a triangle with $\angle BDC = 120°$. Then the X_3 point of $\triangle BDC$ is the center of the equilateral triangle erected outward on side BC.*

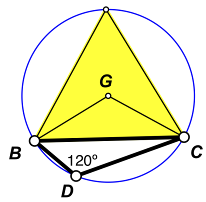


FIGURE 21.

Proof. Let G be the X_3 point of $\triangle BCD$ (Figure 21). It suffices to prove that $\angle CBG = \angle BCG = 30°$. Since G is the circumcenter of $\triangle BCD$, the measure of $\angle BGC$ is equal to the measure of the minor arc $\widehat{BC}$ which is 360° minus the measure of major arc $\widehat{BC}$. This major arc is twice the measure of the inscribed angle, $\angle BDC$, so has measure 240°. Thus, minor arc $\widehat{BC}$ has measure 120° which implies that $\angle BGC = 120°$. Since $GB = GC$, we can conclude that $\angle CBG = \angle BCG = 30°$. □

Theorem 4.9. *Triangles BCD, CAE, and ABF are erected externally on the sides of $\triangle ABC$ so that $\angle BDC = \angle CEA = \angle AFB = 120°$ as shown in Figure 22. If G, H, and I are the X_{110} points of these triangles, then $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$. If the three triangles are erected inward, then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$.*

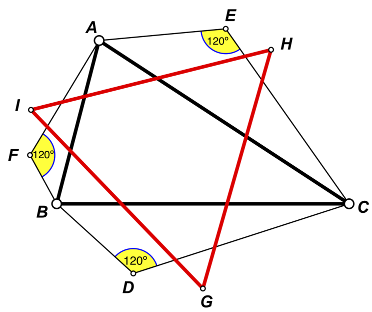


FIGURE 22. X_{110} points

Theorem 4.9 (with the triangles erected outward) follows from the following lemma. The proof is similar for the case when the triangles are erected inward.

Lemma 4.10. *Let ABC be a triangle with $\angle A = 120°$. Then the X_{110} point of $\triangle ABC$ is the center of the equilateral erected outward on side BC (Figure 23).*

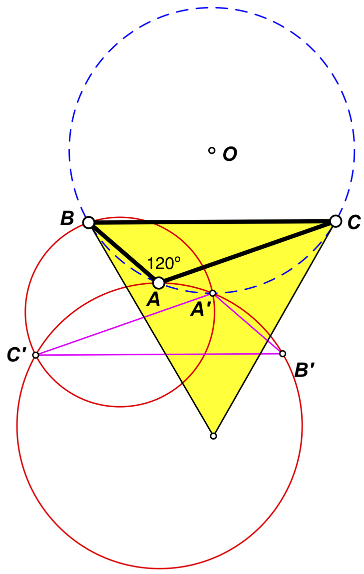


Figure 23.

Proof. Let O be the circumcenter of $\triangle BCD$. Let A', B', and C'be the reflections of O about the sides of $\triangle ABC$ (Figure 17).

Triangle $A'B'C'$ is the image of the medial triangle of ABC under a homothety with center O and ratio 2, so $\triangle ABC \cong \triangle A'B'C'$. Thus, $\angle C'A'B' = 120°$.

Since AC is the perpendicular bisector of OB' and AB is the perpendicular bisector of OC', this means $\angle B'AO = 2\angle CAO$ and $\angle C'AO = 2\angle BAO$. Thus, $\angle B'AO + \angle C'AO = 240°$ and hence $\angle B'AC' = 120°$.

Since $\angle C'A'B' = \angle C'AB' = 120°$, this means that points C', A, B', and A' are concyclic or that A' lies on $\odot C'AB'$.

Thus, $\odot C'AB'$ meets circles $\odot B'CA'$ and $\odot C'BA'$ at A'. But according to property (1) of [8], the circumcircles of triangles $C'A'B'$, $B'CA'$, and $C'BA'$ meet at the X_{110} point of $\triangle ABC$. Consequently, A' is the X_{110} point of $\triangle ABC$. Since A' is the reflection of O about BC, $A'B = A'C$ and $\angle A'BC = \angle A'CB$. But $\angle BA'C = \angle BAC = 120°$, so $\angle A'BC = \angle A'CB = 30°$ and A' is the center of the equilateral triangle erected inward on side BC. □

Our program also discovered similar results when the outer angle is 150°.

Theorem 4.11. *Triangles BCD, CAE, and ABF are erected externally on the sides of $\triangle ABC$ so that $\angle BDC = \angle CEA = \angle AFB = 150°$ as shown in Figure 24. If G, H, and I are the X_{54} points of these triangles, then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$. If the three triangles are erected inward, then $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$.*

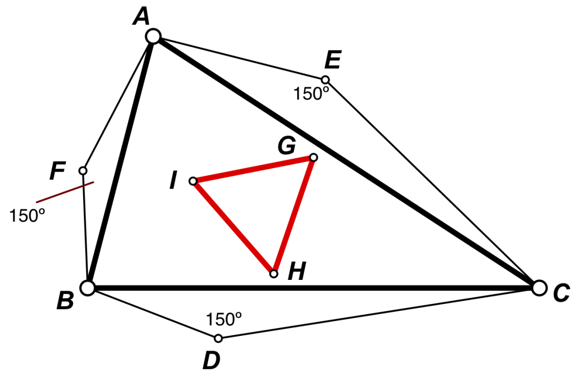


FIGURE 24. X_{54} points

Theorem 4.11 (with the triangles erected outward) follows immediately from the following lemma. The proof is similar for the case when the triangles are erected inward.

Lemma 4.12. *Let ABC be a triangle with $\angle BAC = 150°$. Then the X_{54} point of $\triangle BDC$ is the center of the equilateral triangle erected outward on side BC (Figure 25).*

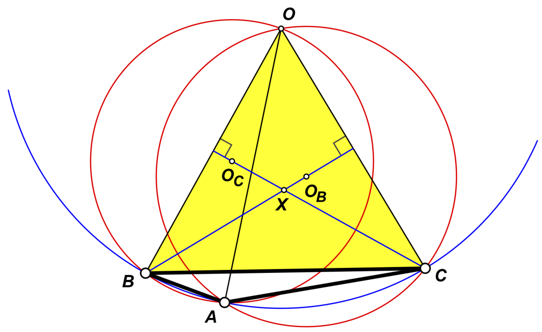


FIGURE 25.

Proof. Let O be the circumcenter of $\triangle ABC$. Note that O is the vertex of the equilateral triangle erected outward on BC because $\angle BOC = m\widehat{BAC} = 360° - 2 \cdot 150° = 60°$. Let O_C be the circumcenter of $\triangle OAB$ and let O_B be the circumcenter of $\triangle OAC$. Since $BO_C = OO_C$, O_C lies on the perpendicular bisector of OB. Since $CO_B = OO_B$, O_B lies on the perpendicular bisector of OC. Let X be the intersection of BO_B and CO_C. Then X is the center of $\triangle OAB$. According to [7], the X_{54} point is the intersection of BO_B and CO_C. Thus, X is the X_{54} point of $\triangle ABC$. □

Theorem 4.13. *Triangles BCD, CAE, and ABF are erected externally on the sides of $\triangle ABC$ so that $\angle BDC = \angle CEA = \angle AFB = 150°$ as shown in Figure 26. If G, H, and I are the X_{195} points of these triangles, then $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$. If the three triangles are erected inward, then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$.*

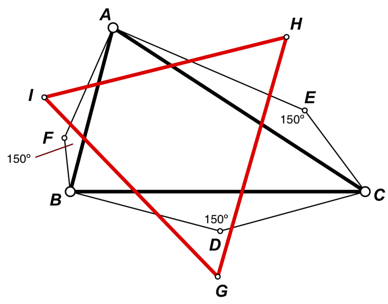
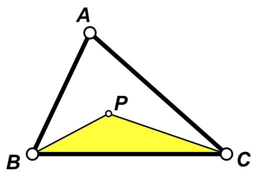


Figure 26. X_{195} points

Proof. According to [9], the X_{195} point of a triangle is the reflection of the X_{54} point about the circumcenter of the triangle. Since the X_{54} points form an equilateral triangle, then so do the X_{195} points. The Napoleon triangles have the same property, so when one is an outer Napoleon triangle, the other will be an inner Napoleon triangle and vice versa. ☐

5. Central Triangles

Let P be any center of $\triangle ABC$. The triangle BCP will be called a *central triangle.*



For any center P of $\triangle ABC$, we can view the three associated central triangles as triangles erected on the sides of $\triangle ABC$.

Our program found the following result.

Theorem 5.1. *Let P be the X_{13} point of $\triangle ABC$. Let BCP, CAP, and ABP be the central triangles associated with P as shown in Figure 27. If G, H, and I are the X_{110} points of these central triangles, then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$.*

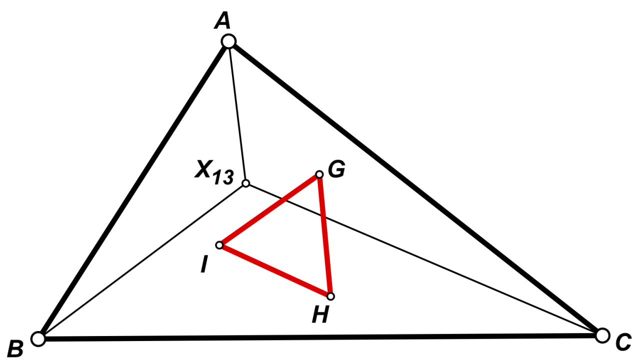


Figure 27. Central triangles for X_{13} and their X_{110} points

Proof. When $P = X_{13}$, it is well known that $\angle BPC = \angle CPA = \angle APB = 120°$. See [6, p. 218]. Thus ABP, BCP, and CAP are triangles erected inward on the sides of $\triangle ABC$ with angles of 120° at P. The result now follows by the inward case of Theorem 4.9. □

Our program found a number of similar results. Instead of listing them each as a separate theorem, we summarize the results in the following table.

Equilateral Triangles Found Using Centers of Central Triangles				
P	**G**	**GHI**	**Perspector**	**Ref**
X_3	X_{13}	X_2	X_{18}	Thm. 2.1
X_3	X_{14}	X_2	X_{17}	Thm. 2.2
X_3	X_{618}	X_{549}	X_3	
X_3	X_{619}	X_{549}	not persp	
X_{13}	X_3	X_2	X_{17}	
X_{13}	X_5	X_{5459}	not persp	[10]
X_{13}	X_{110}	X_2	X_{18}	[2]
X_{13}	X_{125}	X_{5459}	not persp	[3]
X_{13}	X_{477}	X_{5463}	not persp	[4]
X_{14}	X_3	X_2	X_{18}	
X_{14}	X_5	X_{5460}	not persp	[11]
X_{14}	X_{110}	X_2	not persp	[2]
X_{14}	X_{125}	X_{5460}	not persp	[3]
X_{14}	X_{477}	X_{5464}	not persp	[4]

The columns in this table are explained below.

P: specifies which center of $\triangle ABC$ is used for point P.

G: specifies which center of the central triangle was used to obtain the points G, H, and I. Hagos [4] has noted that when $P = X_{13}$ or $P = X_{14}$, $X_{930}[BCD] = X_{477}[BCD]$

GHI: specifies the center of $\triangle ABC$ that coincides with the center of equilateral triangle GHI. Of particular interest are the following facts.

- X_{549} is the midpoint of the line segment joining X_2 and X_3.
- X_{5459} is the midpoint of the line segment joining X_2 and X_{13}.
- X_{5460} is the midpoint of the line segment joining X_2 and X_{14}.
- X_{5463} is the reflection of X_{13} in X_2.
- X_{5464} is the reflection of X_{14} in X_2.

Perspector: specifies the perspector of triangles ABC and GHI expressed as a center relative to $\triangle ABC$. If triangles ABC and GHI are not perspective, the entry reads "not persp".

Ref: If we found the result in the literature, a citation to the bibliography is given. If the result follows from a previous theorem, the theorem number is given.

The following table describes the relationships between the equilateral triangles found and the Napoleon triangles of $\triangle ABC$.

Relationship to Napoleon Triangles of $\triangle ABC$					
P	G	GHI	relationship	homothetic Center	ratio
X_3	X_{13}	X_2	coincides with inner		1
X_3	X_{14}	X_2	coincides with outer		1
X_3	X_{618}	X_{549}	homothetic with outer	X_3	1/2
X_3	X_{619}	X_{549}	homothetic with inner	X_3	1/2
X_{13}	X_3	X_2	coincides with outer		1
X_{13}	X_5	X_{5459}	homothetic with inner	X_{13}	1/2
X_{13}	X_{110}	X_2	coincides with inner		1
X_{13}	X_{125}	X_{5459}	homothetic with outer	X_{13}	1/2
X_{13}	X_{477}	X_{5463}	homothetic with outer	X_{13}	2
X_{14}	X_3	X_2	coincides with inner		1
X_{14}	X_5	X_{5460}	homothetic with outer	X_{14}	1/2
X_{14}	X_{110}	X_2	coincides with outer		1
X_{14}	X_{125}	X_{5460}	homothetic with inner	X_{14}	1/2
X_{14}	X_{477}	X_{5464}	homothetic with inner	X_{14}	2

6. CIRCUMCEVIAN TRIANGLES

Let P be any point inside $\triangle ABC$. If the line AP meets the circumcircle of $\triangle ABC$ again at point D, then segment AD is called a *circumcevian* of $\triangle ABC$. The triangle BCD is called a *circumcevian triangle*. See Figure 28.

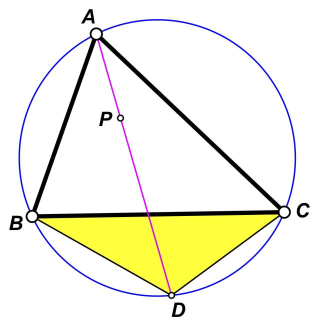


FIGURE 28. Circumcevian triangle associated with Point P.

For any point P inside $\triangle ABC$, we can view the three associated circumcevian triangles as triangles erected on the sides of $\triangle ABC$.

Our program found the following results. The only time we found equilateral triangles was when P is the incenter of $\triangle ABC$. In those cases, the equilateral triangles found coincide with the Napoleon triangles.

Theorem 6.1. *Let P be the incenter of $\triangle ABC$. Let BCD, CAE, and ABF be the circumcevian triangles of point P as shown in Figure 29. If G, H, and I are the X_{13} points of these triangles, then $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$.*

Proof. Since P is the incenter, AD is an angle bisector and thus $\widehat{BD} = \widehat{CD}$ which implies $BD = CD$. Thus, $\triangle BCD$ is an isosceles triangle erected outward on the side BC of $\triangle ABC$. The result then follows from Theorem 2.1. $\square$

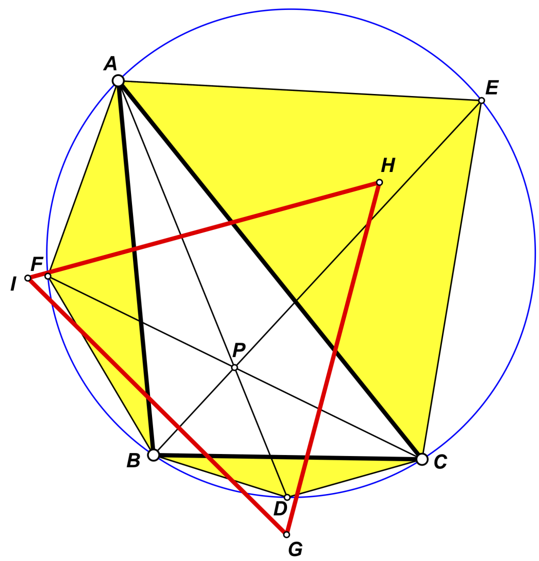


FIGURE 29. Circumcevian triangles and their X_{13} points

Theorem 6.2. *Let P be the incenter of $\triangle ABC$. Let BCD, CAE, and ABF be the circumcevian triangles of point P as shown in Figure 30. If G, H, and I are the X_{14} points of these triangles, then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$.*

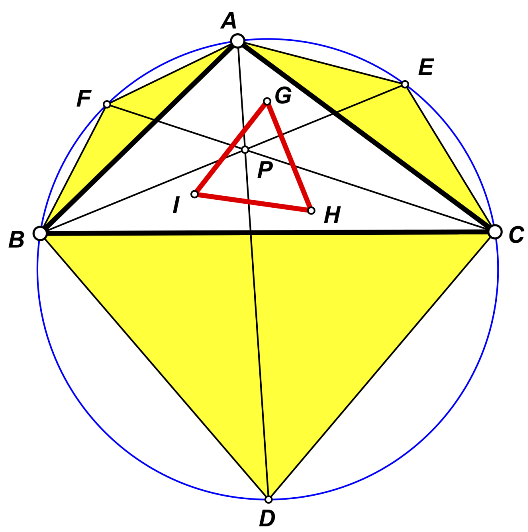


FIGURE 30. Circumcevian triangles and their X_{14} points

Proof. Since P is the incenter, AD is an angle bisector and thus $\widehat{BD} = \widehat{CD}$ which implies $BD = CD$. Thus, $\triangle BCD$ is an isosceles triangle erected outward on the side BC of $\triangle ABC$. The result then follows from Theorem 2.2. $\square$

7. Center Reflection Triangles

Let P be any point in the plane of $\triangle ABC$. Reflect point P about BC to get point D, as shown in Figure 31. Triangles BCD will be called a *center reflection triangle* associated with point P.

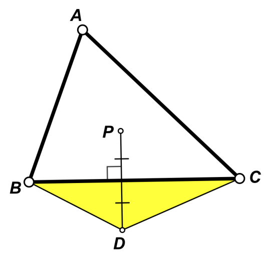


Figure 31. Center Reflection Triangle Associated with Point P.

For any point P inside $\triangle ABC$, we can view the three center reflection triangles as triangles erected on the sides of $\triangle ABC$.

Our program found the following results. The only time we found equilateral triangles was when P is the circumcenter or one of the Fermat points of $\triangle ABC$. In those cases, the equilateral triangles found coincide with the Napoleon triangles.

Theorem 7.1. *Let O be the circumcenter of $\triangle ABC$. Let BCD, CAE, and ABF be the center reflection triangles of point O as shown in Figure 32. If G, H, and I are the X_{13} points of these triangles, then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$.*

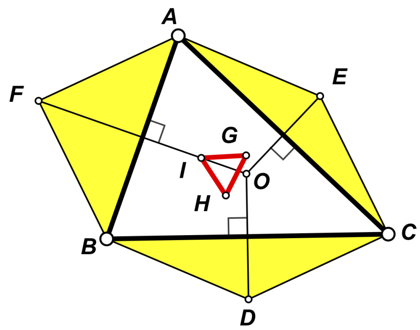


Figure 32. Center Reflection Triangles and their X_{13} points

Theorem 7.2. *Let O be the circumcenter of $\triangle ABC$. Let BCD, CAE, and ABF be the center reflection triangles of point O as shown in Figure 33. If G, H, and I are the X_{14} points of these triangles, then $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$.*

Theorems 7.1 and 7.2 follow from Theorems 2.1 and 2.2 since the erected triangles are isosceles.

Theorem 7.3. *Let P be the X_{13} point of $\triangle ABC$. Let BCD, CAE, and ABF be the center reflection triangles of point P as shown in Figure 34. If G, H, and I are the circumcenters of these triangles, then $\triangle GHI$ is the inner Napoleon triangle of $\triangle ABC$.*

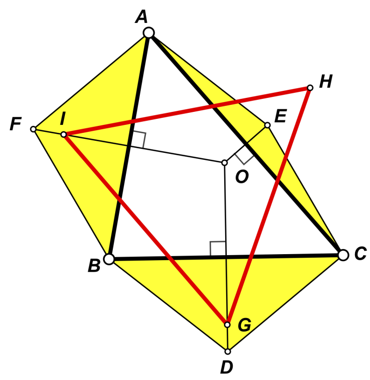


FIGURE 33. Center Reflection Triangles and their X_{14} points

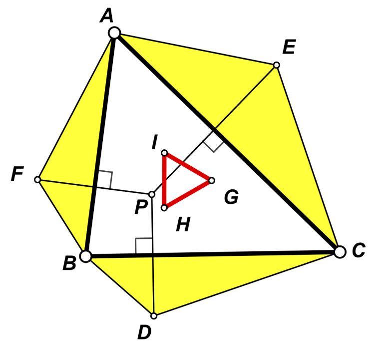


FIGURE 34. Center Reflection Triangles and their circumcenters

Theorem 7.4. *Let P be the X_{14} point of $\triangle ABC$. Let BCD, CAE, and ABF be the center reflection triangles of point P as shown in Figure 35. If G, H, and I are the circumcenters of these triangles, then $\triangle GHI$ is the outer Napoleon triangle of $\triangle ABC$.*

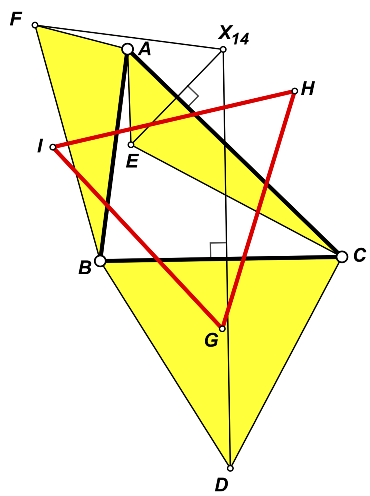


FIGURE 35. Center Reflection Triangles and their circumcenters

Theorems 7.3 and 7.4 follow from properties of the Napoleon triangles such as (in this case) the fact that $\angle APB = \angle BPC = \angle CPA = 120°$.

8. Circlecevian Triangles

Let P be any point inside $\triangle ABC$. If the line AP meets the circle BPC again at point D, then segment AD is called a *circlecevian* of $\triangle ABC$. The triangle BCD is called a *circlecevian triangle*. See Figure 36.

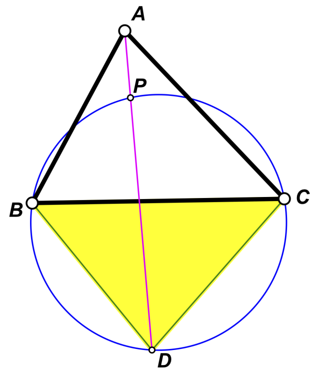


Figure 36. Circlecevian triangle associated with point P.

For any point P inside $\triangle ABC$, we can view the three associated circlecevian triangles as triangles erected on the sides of $\triangle ABC$.

Our program found the following results which were also found by Altıntaş [1].

Theorem 8.1. *Let BCD, CAE, and ABF be the circlecevian triangles of point P inside $\triangle ABC$ as shown in Figure 37. If G, H, and I are the X_{15} points of these triangles, then $\triangle GHI$ is an equilateral triangle.*

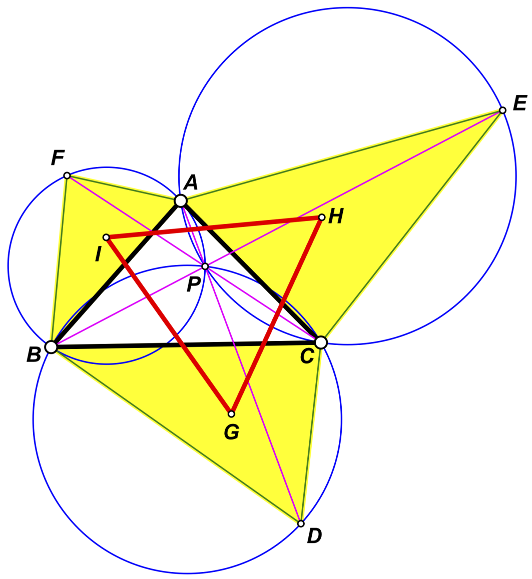


Figure 37. Circlecevian triangles and their X_{15} points

Theorem 8.2. *Let BCD, CAE, and ABF be the circlecevian triangles of point P inside $\triangle ABC$ as shown in Figure 38. If G, H, and I are the X_{16} points of these triangles, then $\triangle GHI$ is an equilateral triangle.*

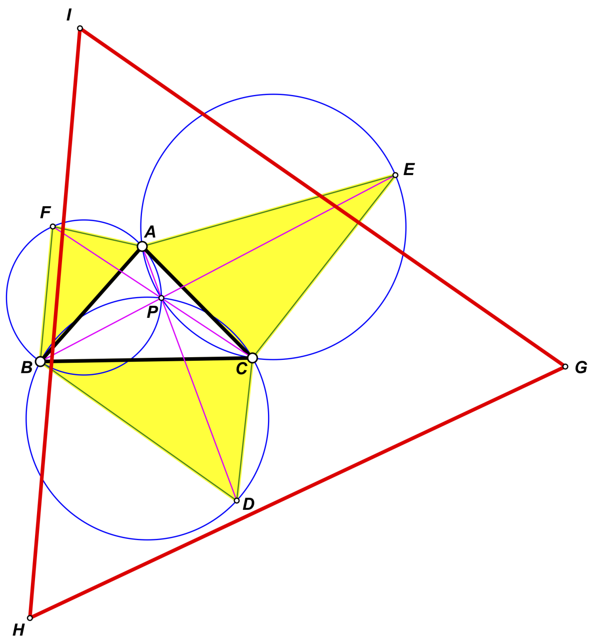


FIGURE 38. Circlecevian triangles and their X_{16} points

9. EQUAL CEVIAL TRIANGLES

There is another way we can generalize Napoleon's Theorem. If BCD, CAE, and ABF are the equilateral triangles erected outward on the sides of $\triangle ABC$ (Figure 39), then it is known that AD, BE, and CF concur at the 1st Fermat point (the X_{13} point) of $\triangle ABC$ and that $AD = BE = CF$. See [6, p. 218].

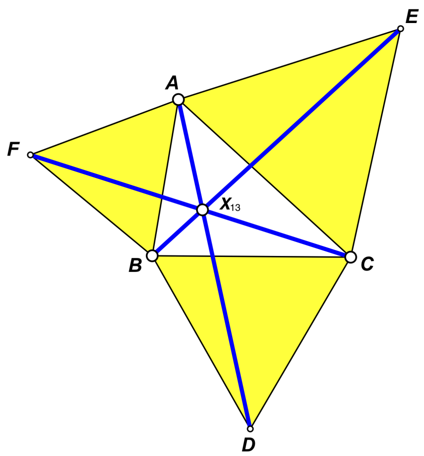


FIGURE 39. Fermat lines related to Napoleon's Theorem

A line from a vertex of a triangle through the X_{13} point of that triangle is called a *Fermat line*. So AD, BE, and CF are equal length Fermat line segments.

If X is some center of $\triangle ABC$, a line from a vertex through X will be called a *cevial line*. If AD, BE, and CF are cevial line segments of equal length through X, then triangles BCD, CAE, and ABF will be called *equal cevial triangles* (Figure 40).

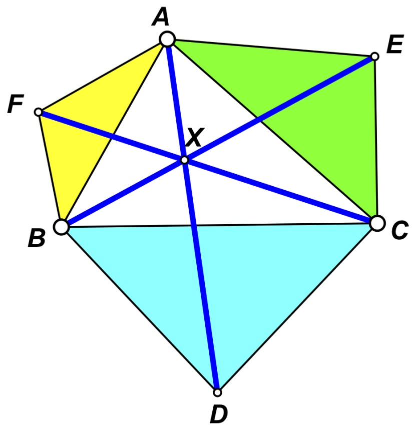


Figure 40. Equal Cevial Triangles

Our program found the following result, which was also found by Oai [12, Theorem 1.9].

Theorem 9.1. *Let X be the 1st (or 2nd) Fermat point of $\triangle ABC$ and let AD, BE, and CF be cevial line segments through X that have the same oriented length. Let G, H, and I be the centroids (X_2 points) of the equal cevial triangles BCD, CAE, and ABF. Then $\triangle GHI$ is an equilateral triangle.*

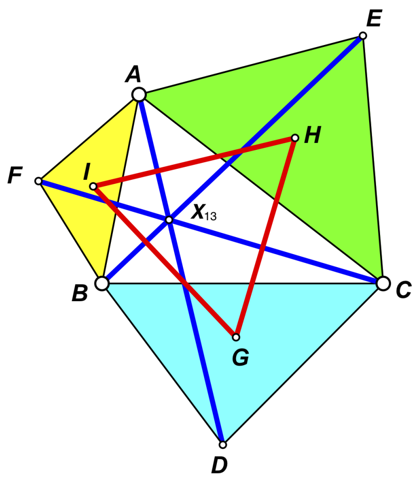


Figure 41. Equal Cevial Triangles and X_{13}

Figure 41 shows the case when $X = X_{13}$ and X lies inside $\triangle ABC$. Figure 42 shows the case when $X = X_{14}$ and X lies outside $\triangle ABC$. (When X is outside $\triangle ABC$, some of the cevial lengths have to be negative, that is, away from X.)

Our program found that the equilateral triangle formed has center $X_2[ABC]$ and is perspective with $\triangle ABC$.

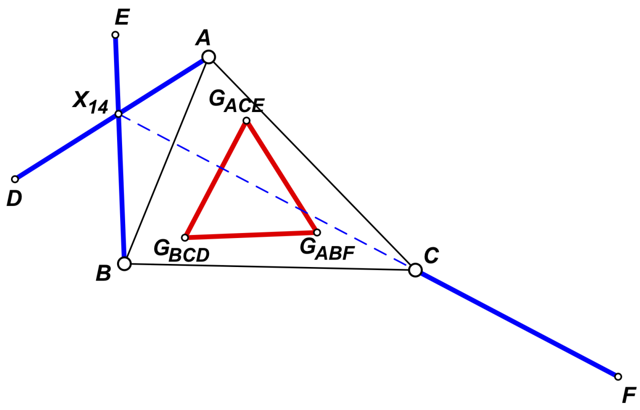


FIGURE 42. $AD = BE = CF$ and X_{14} lying outside $\triangle ABC$

We close with a final bonus result.

Theorem 9.2. *Let X be the 1st (or 2nd) Fermat point of $\triangle ABC$ and let AD, BE, and CF be cevial line segments through X that have the same oriented length. Let G, H, and I be the centroids (X_2 points) of triangles BCD, CAE, and ABF. Let J, K, and L be the centroids of triangles FAE, ECD, and DBF. Then $GKIJHL$ is a regular hexagon.*

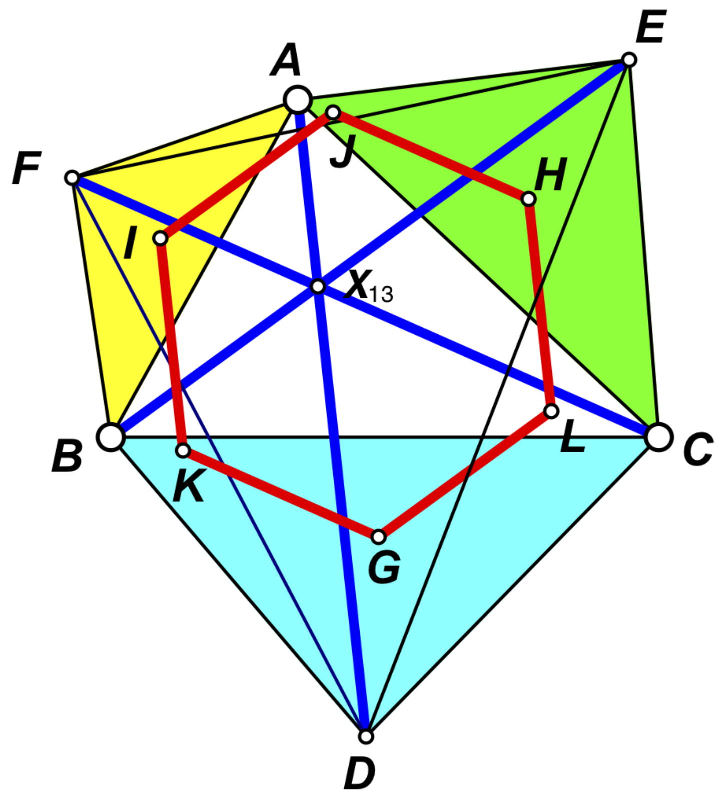


FIGURE 43. Bonus result: A regular hexagon

A proof can be found at [13].

References

[1] Abdilkadir Altıntaş, *Günlüğü item 1574*. Geometry Diary.
https://geometry-diary.blogspot.com/2020/11/1574.html

[2] Abdilkadir Altıntaş, *Message 626*. euclid@groups.io, Geometry Research Mailing List.
https://groups.io/g/euclid/message/626

[3] Abdilkadir Altıntaş, *Message 1416*. euclid@groups.io, Geometry Research Mailing List.
https://groups.io/g/euclid/message/1416

[4] Elias M. Hagos, *Message 1419*. euclid@groups.io, Geometry Research Mailing List.
https://groups.io/g/euclid/message/1419

[5] Huseyin Demir, Solution to Problem E2122, *American Mathematical Monthly*, 76(1969)833.
https://doi.org/10.2307/2317899

[6] Roger Arthur Johnson, *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*. Houghton Miflin, Boston: 1929.
http://books.google.com/books?id=KVdtAAAAMAAJ

[7] Clark Kimberling, *X(54)*, Encyclopedia of Triangle Centers.
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X54

[8] Clark Kimberling, *X(110): Focus of the Kiepert Parabola*, Encyclopedia of Triangle Centers.
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X110

[9] Clark Kimberling, *The Kosnita Point X(195)*, Encyclopedia of Triangle Centers.
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X195

[10] Clark Kimberling, *X(5459), the midpoint of* X_2 *and* X_{13}, Encyclopedia of Triangle Centers.
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X5459

[11] Clark Kimberling, *X(5460), the midpoint of* X_2 *and* X_{14}, Encyclopedia of Triangle Centers.
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X5460

[12] Dao Thanh Oai, *Some Equilateral Triangles Perspective to the Reference Triangle ABC*, International Journal of Computer Discovered Mathematics, 3(2018)88–96.
http://www.journal-1.eu/2018/Dao-Perspective-Triangles.pdf

[13] Dao Thanh Oai, *Napoleon's Hexagon* at Cut-The-Knot.
https://www.cut-the-knot.org/m/Geometry/NapoleonsHexagon.shtml

[14] Stanley Rabinowitz, Problem E2122, *American Mathematical Monthly*, 75(1968)898.
https://doi.org/10.2307/2314357

[15] Stanley Rabinowitz, *Post SR30*, Plane Geometry Research Facebook Group, Feb. 1, 2021.
https://www.facebook.com/groups/2008519989391030/permalink/2823537951222559/

[16] Yung-Chow-Wong, Some Properties of the Triangle, *American Mathematical Monthly*, 48(1941)530–535.
https://doi.org/10.2307/2303388

[17] Paul Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, December 2012.
http://math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry121226.pdf