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Linear Relationships Between Squares of Cevian Lengths

Stanley Rabinowitz
545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net
web: http://www.StanleyRabinowitz.com/

Abstract. A cevian is a line segment joining the vertex of a triangle and a point on the opposite side. Well-known cevians are medians, angle bisectors, and altitudes. We consider various cevians passing through named triangle centers such as the Gergonne point and Nagel point. We show how a computer can be used to discover, not just prove, identities involving the squares of the lengths of these cevians.

Keywords. triangle geometry, Gergonne cevian, Nagel cevian, identities, Mathematica, computer-discovered mathematics.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

Let m_a denote the length of the median to side a of a triangle with sides of lengths a, b, and c. Similar notation is used for the other two medians. The identity

$$4(m_a^2 + m_b^2 + m_c^2) = 3(a^2 + b^2 + c^2) \tag{1}$$

is well known [1, p. 70]. It is the purpose of this paper to show how similar relationships involving the sum of the squares of the lengths of other cevians can be discovered, not just proven, by computer. We will find a number of such relationships, believed to be new.

A *cevian* is a line segment joining the vertex of a triangle and a point on the opposite side. Let r, R, and s denote the inradius, circumradius, and semiperimeter of the triangle. We use the notation

$$\sum f(a, b, c)$$

to denote the cyclic sum $f(a, b, c) + f(b, c, a) + f(c, a, b)$. Thus, equation (1) can be written as

$$4 \sum m_a^2 = 3 \sum a^2.$$

A cevian through the Gergonne point of a triangle is called a *Gergonne cevian* and a cevian through the Nagel point of a triangle is called a *Nagel cevian.* The lengths of the Gergonne and Nagel cevians to side a of a triangle will be called g_a and n_a, respectively. The corresponding lengths to sides b and c are named similarly.

Let X_n denote the n-th named triangle center in the Encyclopedia of Triangle Centers [3]. A cevian through X_n shall be called an X_n-cevian. The line segment from a vertex to the point X_n shall be called an X_n-spoke. In this paper, we will look for linear relationships between the squares of cevians and spokes associated with triangle centers.

2. THE RESULTS

The following results are believed to be new.

Theorem 1. *The following identities hold for all triangles.*

$$\sum g_a^2 + \sum n_a^2 + 2s^2 = 4\sum m_a^2$$
$$3\sum g_a^2 + 3\sum n_a^2 + 3\sum bc = 10\sum m_a^2$$
$$3\sum g_a^2 + 3\sum n_a^2 + 6r^2 + 24R^2 = 8\sum m_a^2$$

For the following theorems, we need some additional notation. Let $d(P,Q)$ denote the distance between points P and Q. In a fixed triangle ABC, let $y_n(a) = d(A, X_n)$. That is, $y_n(a)$ is the length of the spoke from A to X_n. For brevity, we will omit the "(a)" and just write $y_n = y_n(a)$.

Theorem 2. *The following identities hold for all triangles.*

$$3\sum y_2^2 + \sum y_8^2 = 4\sum y_1^2$$
$$4\sum y_1^2 + 3\sum y_2^2 = 4\sum y_{10}^2$$
$$\sum y_8^2 + 4\sum y_{10}^2 = 5\sum y_1^2$$
$$15\sum y_1^2 + \sum y_8^2 = 16\sum y_{10}^2$$
$$3\sum y_2^2 + \sum y_4^2 = 4\sum y_3^2$$
$$3\sum y_2^2 + \sum y_3^2 = 4\sum y_5^2$$
$$15\sum y_2^2 + \sum y_4^2 = 16\sum y_5^2$$
$$\sum y_4^2 + 4\sum y_5^2 = 5\sum y_3^2$$
$$3\sum y_2^2 + \sum y_7^2 = 4\sum y_9^2$$

Theorem 3. *The following identities hold for all triangles.*

$$
\begin{aligned}
4\sum y_1^2 + \sum y_4^2 &= 4\sum y_3^2 + \sum y_8^2\\
4\sum y_1^3 + \sum y_3^2 &= 4\sum y_{10}^2 + \sum y_4^2\\
4\sum y_1^2 + \sum y_3^2 &= 4\sum y_5^2 + \sum y_8^2\\
4\sum y_5^2 + \sum y_1^2 &= 4\sum y_{10}^2 + \sum y_3^2\\
4\sum y_1^2 + \sum y_7^2 &= 4\sum y_9^2 + \sum y_8^2\\
4\sum y_9^2 + \sum y_1^2 &= 4\sum y_{10}^2 + \sum y_7^2\\
4\sum y_3^2 + \sum y_7^2 &= 4\sum y_9^2 + \sum y_4^2\\
4\sum y_6^2 + \sum y_7^2 &= 4\sum y_9^2 + \sum y_3^2\\
5\sum y_1^2 + 16\sum y_5^2 &= 20\sum y_{10}^2 + \sum y_4^2\\
5\sum y_3^2 + 16\sum y_{10}^2 &= 20\sum y_5^2 + \sum y_8^2\\
5\sum y_7^2 + 16\sum y_{10}^2 &= 20\sum y_9^2 + \sum y_8^2
\end{aligned}
$$

There are also some relationships not involving symmetric sums.

Theorem 4. *The following identities hold for all triangles.*

$$
\begin{aligned}
9y_2^2 + y_4^2 &= 8y_5^2 + 2y_3^2\\
9y_2^2 + y_8^2 &= 8y_{10}^2 + 2y_1^2
\end{aligned}
$$

3. The Proofs

The proofs of these results are straightforward using trilinear coordinates. Set up a trilinear coordinate system with $\triangle ABC$ as the reference triangle, so that $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $C = (0, 0, 1)$. Let P be an arbitrary point in the plane other than A and let AP meet BC at D. Then $D = (0, q, r)$.

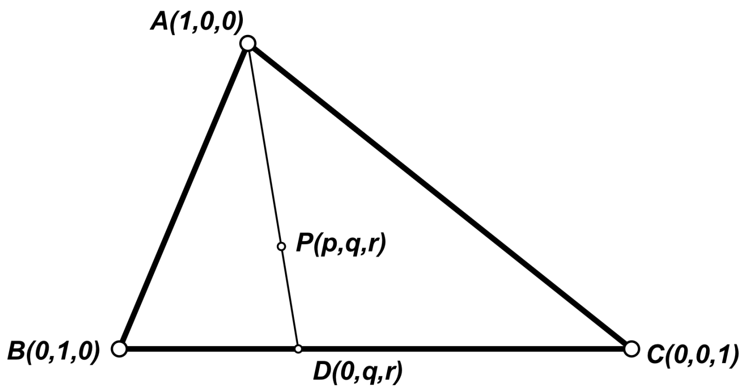


Figure 1. Trilinear Coordinates

Theorem 5 (Cevian and Spoke Lengths). *Let P be a point in the plane of $\triangle ABC$ with trilinear coordinates (p, q, r). Then*

$$AD = \frac{\sqrt{b^2r(q+r) + c^2q(q+r) - a^2qr}}{q+r}$$

and

$$AP = \frac{\sqrt{b^2r(q+r) + c^2q(q+r) - a^2qr}}{p+q+r}.$$

Proof. These results follow from the formula for the distance between two points in trilinear coordinates [2, formula 9]. □

The trilinear coordinates for the various triangle centers can be found in [3]. These can be expressed in terms of a, b, and c, the lengths of the sides of the reference triangle. The values of r, R, and s are also well known [1]:

$$\begin{aligned} r &= K/s \\ R &= abc/(4K) \\ s &= (a+b+c)/2 \end{aligned}$$

where

$$K = \sqrt{s(s-a)(s-b)(s-c)}.$$

Using these values and Theorem 5, Theorems 1–4 can be proven by substituting these values in for y_i, r, R, and s and verifying that the resulting formula is true using algebraic simplification.

4. A GEOMETRIC PROOF

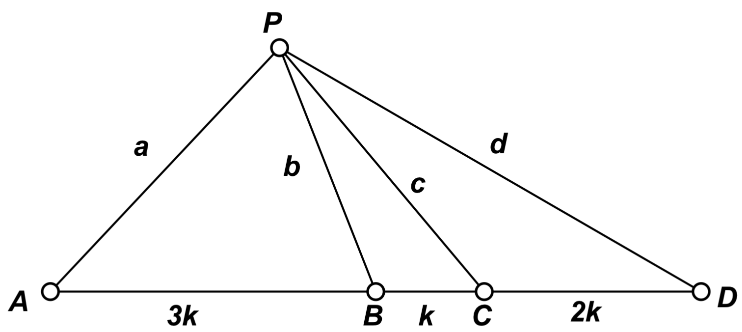
Some of the results can also be proven geometrically.

Lemma 6 ([4]). *Let A, B, C, and D be four collinear points such that*

$$AB : BC : CD = 3 : 1 : 2.$$

Let P be a point in the plane and let $PA = a$, $PB = b$, $PC = c$, and $PD = d$. Then

$$a^2 + 9c^2 = 8b^2 + 2d^2.$$



Proof. By Stewart's Theorem [1, p. 152], we have

$$a^2(k) + c^2(3k) = b^2(4k) + (3k)(k)(4k)$$

and

$$b^2(2k) + d^2(k) = c^2(3k) + (k)(2k)(3k).$$

Eliminating k from these two equations gives the desired result. □

Lemma 7. *Let X_n denote the n-th named triangle center in the Encyclopedia of Triangle Centers* [3]. *Then X_3, X_2, X_5, and X_4 colline and X_1, X_2, X_{10}, and X_8 colline. Furthermore,*

$$X_3X_2 : X_2X_5 : X_5X_4 = 2 : 1 : 3$$

and

$$X_1X_2 : X_2X_{10} : X_{10}X_8 = 2 : 1 : 3.$$

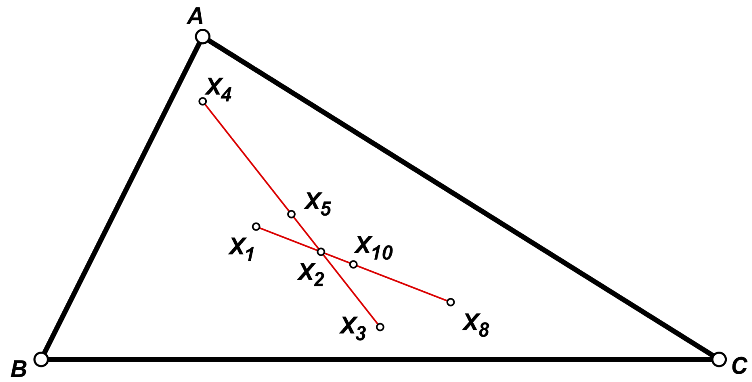


Figure 2. Collinear Centers

Proof. These are well-known properties of a triangle's Euler line [5] and its Nagel line [6]. □

Theorem 4 now follows from Lemma 6 and Lemma 7.

5. The Discovery Process

Once an identity involving the lengths of cevians is known, the proof is relatively simple using trilinear coordinates. The interesting issue is how to discover such results.

Suppose we want to look for a linear relationship between some or all of the $\{y_i\}$, $i = 1, 2, ..., 10$.

The program Mathematica has a function called `FindIntegerNullVector` that finds linear relationships between a set of numbers. According to [7], it uses Storjohann's variant of the Lenstra–Lenstra–Lovasz lattice reduction algorithm and is reasonably efficient. Unfortunately, y_i is an expression in terms of a, b, and c. The function `FindIntegerNullVector` will not work with non-constant expressions.

We can use the following technique. Substitute numerical values for a, b, and c in the expressions and then use `FindIntegerNullVector` to find a linear relationship between the numerical values. If the values chosen are random enough, the linear relationship will also hold for the expressions. In particular, if we choose

$$a = \pi, \quad b = e, \quad \text{and} \quad c = \sqrt[3]{11},$$

where e is the base of natural logarithms, then it is extremely unlikely that there is a simple polynomial relationship between a, b, and c. We choose these three values because they satisfy the triangle inequality.

In Mathematica, we let `y[i]` be the square of the length of y_i, and issue the following commands.

```
vector = Table[y[i], {i, 1, 10}];
FindIntegerNullVector[vector/.{a->Pi, b->E, c->11^(1/3)}]
```

Mathematica responds with

$\{-2, 0, 2, -1, 8, 0, 0, 1, 0, -8\}$

which tells us that it has found the linear relationship

$$-2y_1^2 + 2y_3^2 - y_4^2 + 8y_5^2 + y_8^2 - 8y_{10}^2 = 0.$$

We have thus discovered a new result.

Theorem 8. *The following identity holds for all triangles.*

$$2y_3^2 + 8y_5^2 + y_8^2 = 2y_1^2 + y_4^2 + 8y_{10}^2$$

The result is easily proven using the expressions for the y_i and a little algebra. It can also be derived by eliminating y_2 from the two equations in Theorem 4.

The results of Theorems 1–4 were discovered using the same technique. Since `FindIntegerNullVector` returns only the first linear relationship it finds, in some cases we simplified the discovery process by applying `FindIntegerNullVector` to all 3- or 4-element subsets of the $\{y_i\}$, $i = 1, 2, ..., 10$.

Additional terms can be added to the vector. For example, if we remove y_1 and y_3 from the vector and add the expression a^2 to the end of the vector, when we call `FindIntegerNullVector`, Mathematica responds with

$\{-18, -1, 16, 0, 0, 0, 0, 0, 1\}$

giving us the following interesting result.

Theorem 9. *The following identity holds for all triangles.*

$$18y_2^2 + y_4^2 = 16y_5^2 + a^2$$

Many more interesting identities can be found using this technique. One final example involving symmetric sums:

Theorem 10. *The following identity holds for all triangles.*

$$2\sum y_8^2 + 5s^2 = 12\sum y_{10}^2 + 15r^2$$

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