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Median-orthologic Simplexes

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Abstract. We generalize the notation of *median-orthologic* triangles in two-dimensional Euclidean geometry to higher dimension.

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The *median-orthology* relationship between two triangles in two-dimensional geometry was proposed in [1]. This triangle relationship was discovered when the author played with the computer software "GeoGebra".

Two triangles are called *median-orthologic* if three lines, respectively drawn from the vertices of the first triangle and perpendicular to the corresponding medians of the second triangle, are concurrent. In this case, reciprocally, three lines, respectively drawn from the vertices of the second triangle and perpendicular to the corresponding medians of the first triangle, are also concurrent. Two concurrent points are then called the *median-orthologic* centers. See figure for an illustration.

In this paper, we generalize this concept to higher dimension. Let $n \geq 2$ be a integer and consider two $n-dimensional$ non-degenerate simplexes $\mathbf{A} = A_1A_2...A_{n+1}$ and $\mathbf{B} = B_1B_2...B_{n+1}$ in the *n-dimensional* Euclidean space $\mathcal{E}^n$. Let G_a, G_b be respectively the centroids of $\mathbf{A}$ and $\mathbf{B}$.

Let a_i, $(i = 1, 2, .., n+1)$, be the $n-dimensional$ hyperplanes respectively containing the vertices A_i of simplex $\mathbf{A}$ and perpendicular to the median line B_iG_b of simplex $\mathbf{B}$. Similarly, let b_i, $(i = 1, 2, .., n+1)$, be the $n-dimensional$ hyperplanes respectively containing the vertex B_i of simplex $\mathbf{B}$ and perpendicular to the median line A_iG_a of simplex $\mathbf{A}$.

Theorem. *The following statements are equivalent:*

(1) $a_1, a_2, .., a_{n+1}$ *intersect at a single point.*
(2) $b_1, b_2, .., b_{n+1}$ *intersect at a single point.*

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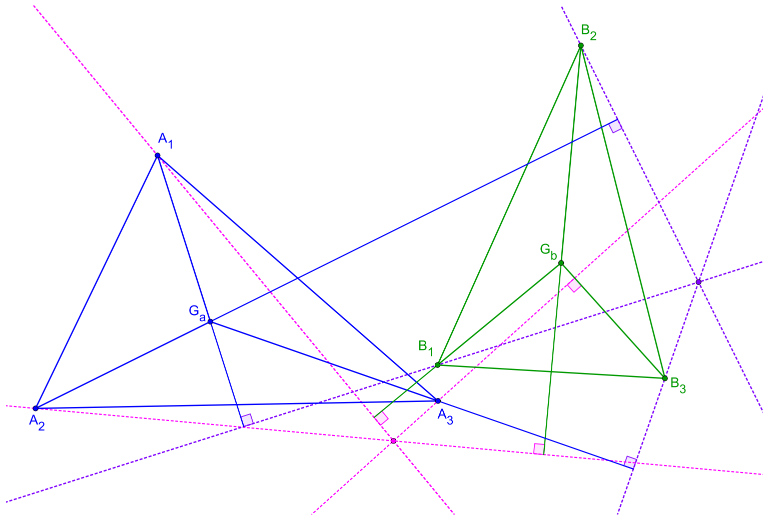

FIGURE 1. Example of a pair of *median-orthologic* triangles $\triangle A_1A_2A_3$, $\triangle B_1B_2B_3$ with centroids G_a, G_b. The lines a_1, a_2, a_3 passing through A_1, A_2, A_3 respectively perpendicular to G_bB_1, G_bB_2, G_bB_3 are concurrent. The lines b_1, b_2, b_3 passing through B_1, B_2, B_3 respectively perpendicular to G_aA_1, G_aA_2, G_aA_3 are also concurrent.

(3) $\sum_{1\le i,j\le n+1} \overrightarrow{A_iA_j} \cdot \overrightarrow{B_iB_j} = 0.$

Here "·" stands for the dot-product.

Definition. *Two simplexes satisfying these statements in Theorem as two* median-orthologic *simplexes. This relation between two simplexes is called* median-orthology. *Two concurrent points are called* median-orthologic centers.

To prove the Theorem , we will take the same steps as in [1]. We need the following key Lemma.

Lemma. *Let M be an arbitrary point on $\mathcal{E}^n$. Then we have the following identities, called here the dot-product identity:*

$$\sum_{1\le i\le n+1} \overrightarrow{MA_i} \cdot \overrightarrow{G_bB_i} = \sum_{1\le i\le n+1} \overrightarrow{MB_i} \cdot \overrightarrow{G_aA_i} = \frac{1}{2(n+1)} \sum_{1\le i,j\le n+1} \overrightarrow{A_iA_j} \cdot \overrightarrow{B_iB_j}.$$

Proof. [of the Lemma]

As G_b is the centroid of the $\mathbf{B}$, we have $\overrightarrow{G_bB_i} = \frac{1}{n+1}\sum_{1\le j\le n+1} \overrightarrow{B_jB_i}$. Consequently,

$$\begin{aligned}
\sum_{1\le i\le n+1} \overrightarrow{MA_i} \cdot \overrightarrow{G_bB_i} &= \frac{1}{n+1} \sum_{1\le i\le n+1} \overrightarrow{MA_i} \cdot \Big(\sum_{1\le j\le n+1} \overrightarrow{B_jB_i}\Big) \\
&= \frac{1}{2(n+1)} \sum_{1\le i,j\le n+1} \left(\overrightarrow{MA_i} \cdot \overrightarrow{B_jB_i} + \overrightarrow{MA_j} \cdot \overrightarrow{B_iB_j}\right) \\
&= \frac{1}{2(n+1)} \sum_{1\le i,j\le n+1} \overrightarrow{A_iA_j} \cdot \overrightarrow{B_iB_j}.
\end{aligned}$$

□

Proof. [of the Theorem]

We only need to prove (1) $\Leftrightarrow$ (3), as by the same argument (2) $\Leftrightarrow$ (3) and the conclusion follows.

Assume that (1) is true, i.e the $n-$*dimensional* hyperplanes b_i intersect at a point M. Since for each i, $MB_i \perp G_aA_i$ we have $\overrightarrow{MA_i} \cdot \overrightarrow{G_aA_i} = 0$. By the dot-product identity we obtain (3).

Now assume that (3) is true, i.e $\sum_{1 \leq i,j \leq n+1} \overrightarrow{A_iA_j} \cdot \overrightarrow{B_iB_j} = 0$. By the dot-product identity we have $\sum_{1 \leq i \leq n+1} \overrightarrow{MB_i} \cdot \overrightarrow{G_aA_i} = 0$. Since $n+1$ vertices A_i of $\mathbf{A}$ do not lie on the same $n-$*dimensional* hyperplane, any two different $n-$*dimensional* hyperplanes among $b_1, b_2, .., b_{n+1}$ are not parallel. Thus, $b_1, b_2, ..., b_n$ intersects at a single point M. For each $i \in \{1, 2, ..., n\}$, $\overrightarrow{MA_i} \cdot \overrightarrow{G_aA_i} = 0$ as $MB_i \perp G_aA_i$. By $\sum_{1 \leq i \leq n+1} \overrightarrow{MB_i} \cdot \overrightarrow{G_aA_i} = 0$ we obtain $\overrightarrow{MB_{n+1}} \cdot \overrightarrow{G_aA_{n+1}} = 0$ or $MB_{n+1} \perp G_aA_{n+1}$. Thus, $M \in b_{n+1}$ and we obtain (1).

$\square$

References

[1] Vu, T.T. (2021). 105.11 Median-parallelogic and median-orthologic triangles. The Mathematical Gazette, 105(562), 136-139. doi:10.1017/mag.2021.24