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PONCELET-GERGONNE CIRCLE OF A TRIANGLE, MOVING BETWEEN TWO FIXED CIRCLES

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Abstract: According to a remarkable theorem of the French mathematician Poncelet, if two conics are located in a plane in such a way, that an inscribed polygon exists in one of them, which is circumscribed with respect to the other one, then each point from any of the two conics generates an inscribed-circumscribed polygon for both of them. Particularly, two circles may be located in the plane in such a way, that one of them is the circumcircle of a triangle while the other one is its incircle. In connection with the configuration when the triangle is moving between the two circles, loci could be considered describing various notable points in the plane of the triangle. What is considered in the sequel is the locus describing Gergonne point in the plane of the moving triangle. It turns out that the locus is a circle with center on the central line of the two fixed circles. The locus itself is named Poncelet-Gergonne circle.
Keywords: triangle, incircle, circumcircle, Gergonne point, Poncelet theorem, GSP

1. Introduction. A remarkable Poncelet theorem is well-known in Euclidean geometry, a particular case of which is asserted in the following

Theorem. *If the circles Γ and ω are located in the plane in such way, that a triangle exists, which is inscribed in Γ and circumscribed with respect to ω, then:*

1) each point from Γ is a vertex of unique triangle, which is inscribed in Γ and circumscribed with respect to ω;

2) each point from ω is tangent point on a side of the unique triangle, which is inscribed in Γ and circumscribed with respect to ω.

Let $\Gamma(O,R)$ and $\omega(J,r)$ be two non-concentric circles verifying the above theorem. It follows from the first item of the theorem, that if A is an arbitrary point on Γ, then there exist unique points B and C on this circle for which the triangle ABC is circumscribed with respect to ω (Fig. 1). If the point A moves on Γ (occupying locations A_1, A_2, … on Γ), then the triangle ABC moves between the circles Γ and ω (occupying locations $A_1B_1C_1$, $A_2B_2C_2$, …), always being inscribed in Γ and circumscribed with respect to ω (Fig. 1). Along the movement an arbitrary point P, which is connected with ΔABC in some way, also moves together with the triangle (P occupies locations P_1, P_2, … together with the corresponding triangles $A_1B_1C_1$, $A_2B_2C_2$, …) (Fig. 1), at the same time describing a determined trajectory in the plane of the triangles. Thus, a task arises to determine the trajectories of some notable points of the triangle when it moves between the circles Γ and ω in the described manner. Loci are considered in [1] regarding the gravity centre, the orthocentre, Nagel point and other points in the plane of the triangle in motion. As shown in [1], these points describe circles with centers on the central line OJ, while their radii depend on the radii R and r of Γ and ω, respectively.

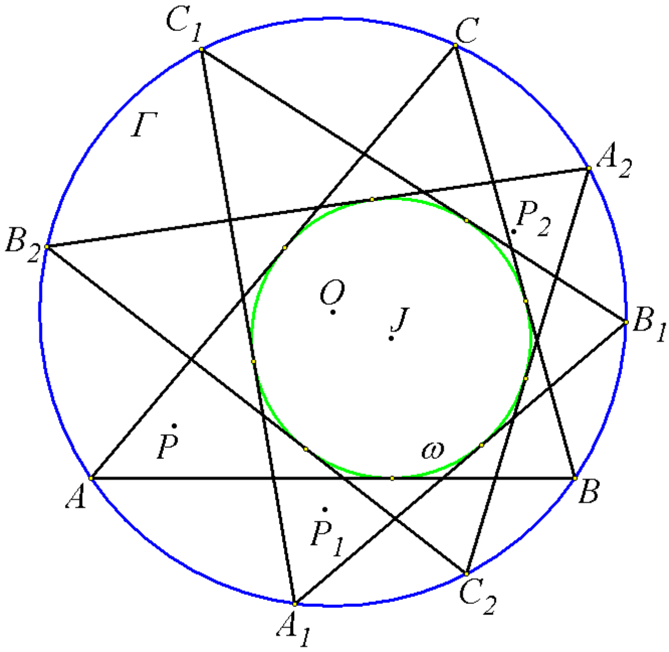

Fig. 1

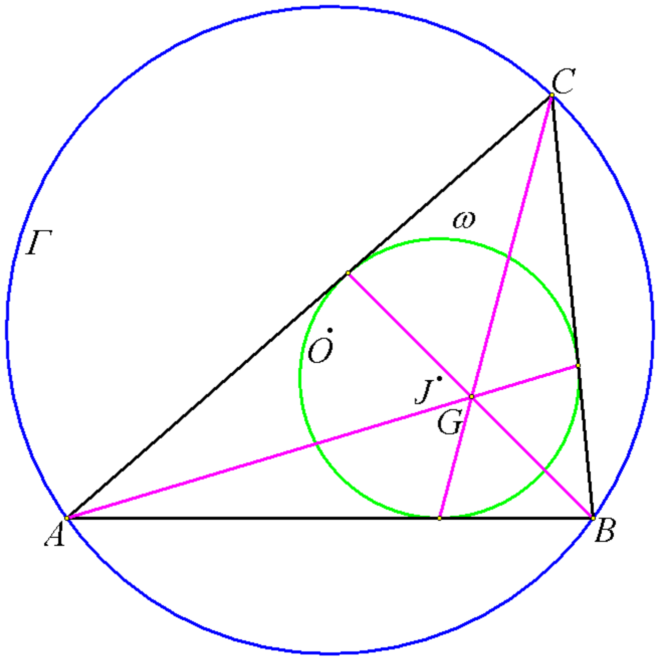

Fig. 2

Another notable point of the triangle ABC is its Gergonne point. This point is the intersection of the lines connecting the vertices of the triangle and the tangent points of its incircle ω to the opposite sides of the corresponding vertices (Fig. 2). If r_a, r_b and r_c are the radii of the excircles of ΔABC, which are tangent to the sides BC, CA and AB, respectively, then the following vector equality is satisfied

(1) $$\overrightarrow{OG}=\frac{r_a\overrightarrow{OA}+r_b\overrightarrow{OB}+r_c\overrightarrow{OC}}{r_a+r_b+r_c}.$$

It will be curious to establish the locus of Gergonne point when ΔABC moves between the fixed circles Γ and ω. Observations with the software Geometer's sketchpad (GSP) show that along the motion of ΔABC between the circles Γ and ω Gergonne point describes a circle $k(G)$ with center on the central line OJ of the fixed circles Γ and ω. As a result of the observations, we find reasons to formulate the following:

Assertion 1. *If G is Gergonne point of the moving triangle ABC between the circles Γ and ω триъгълник, it describe a circle $k(G)$ with center on the central line of Γ and* ω (Fig. 3).

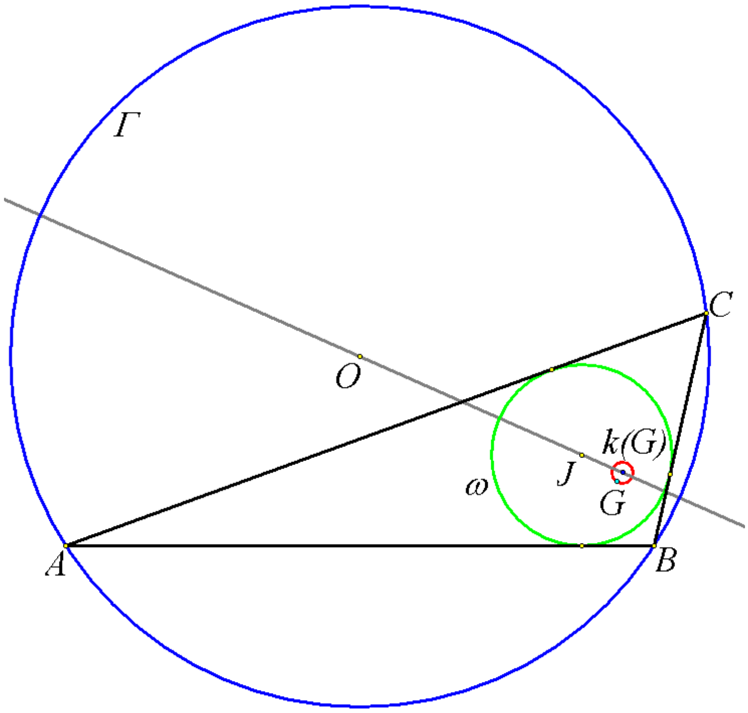

Fig. 3

The circle corresponding to assertion 1 will be called Poncelet-Gergonne circle. The radius of Poncelet-Gergonne circle $k(G)$ is guessed after GSP observations. However, this is a rather difficult task compared with the cases for the circles that are described in [1]. The establishment that the locus under observation is the circle $k(G)$ should not be realized by the means from [1]. We need another approach. For the purpose we will consider geometric objects in the complex plane with respect to a coordinate system with center O and real axis OJ. As usually, the points will be denoted with capital letters, while their affixes – with corresponding small letters. It is assumed for convenience that $R=1$.

One could apply the well-known Euler formula $OJ^2=R^2-2Rr$ for the distances between the centers O and J, also between the centers of Γ and ω. From here, taking into account that $\overline{j}=j$, we obtain:

(2) $$j^2=1-2r.$$

Further, concerning the vertices of the moving tringle ABC we should determine their affixes. We will use the equality (1) for the affix g of Gergonne point G. Accordingly, we need the radii r_a, r_b and r_c. Their values could be determined if the centers of the corresponding excircles are known. On the other hand, the centers are the intersection points of the angular bisectors of ΔABC. The plan for the determination of the point G with respect to the introduced coordinate system is realized in the following several items.

2. Determination of the vertices of the triangle. Let A and W be arbitrary points on the circle Γ. The equation of the line AW is $AW:(\overline{a}-\overline{w})z-(a-w)z-(\overline{a}w-a\overline{w})=0$. Since $\overline{a}=\frac{1}{a}$ and $\overline{w}=\frac{1}{w}$, it follows from the equation of AW that

(3) $$\overline{z}=\frac{a+w-z}{aw}.$$

The equation of ω is $\omega:(z-j)(\overline{z}-\overline{j})=r^2$. Accountingly, for (2) and (3) we obtain the next equation with respect to z. Consequently

$$z^2-\left[(wj-1)a-w-j\right]z+\left[(r^2+2r-1)w+j\right]a+wj=0.$$

The discriminant of this equation is

$$d_0=\left[(1-2r)a^2-2ja+1\right]w^2-2\left[ja^2+2(r^2+r-1)a+j\right]w+a^2-2ja-2r+1.$$

The line AW is tangent to ω only in the case $d_0=0$. Using the equality (2), in the case $d_0=0$ we obtain the values of w:

$$w_1=\frac{ja^2+2(r^2+r-1)a+j+2rd}{(1-2r)a^2-2ja+1},\ w_2=\frac{ja^2+2(r^2+r-1)a+j-2rd}{(1-2r)a^2-2ja+1},$$

where

(4) $$d=\sqrt{a\left[ja^2+(r^2+2r-2)a+j\right]}.$$

Assuming that $b=w_1$ and $c=w_2$, we obtain the following equalities for the affixes of the vertices B and C

(5) $$b=\frac{ja^2+2(r^2+r-1)a+j+2rd}{(1-2r)a^2-2ja+1},\ \overline{b}=\frac{ja^2+2(r^2+r-1)a+j-2rd}{a^2-2ja-2r+1},$$

(6) $$c=\frac{ja^2+2(r^2+r-1)a+j-2rd}{(1-2r)a^2-2ja+1},\ \overline{c}=\frac{ja^2+2(r^2+r-1)a+j+2rd}{a^2-2ja-2r+1}.$$

Thus, the affixes of the vertices B and C of ΔABC are determined as functions of the moving point A on Γ.

3. Determination of the angular bisectors of the triangle. The internal angular bisector l_a from the vertex A of ΔABC is determined by the points A and J (Fig. 4). Thus, the corresponding equation is:

(7) $$l_a:(1-ja)z-(a-j)a\overline{z}+ja^2-j=0.$$

The external angular bisector l'_a from the vertex A of ΔABC is perpendicular to l_a (Fig. 4). For this reason each point Z on the line under consideration satisfies the equality $(z-a)(\overline{j}-\overline{a})+(\overline{z}-\overline{a})(j-a)=0$. From here we obtain the equation of the line l'_a:

(8) $$l'_a:(1-ja)z+(a-j)a\overline{z}+ja^2-2a+j=0.$$

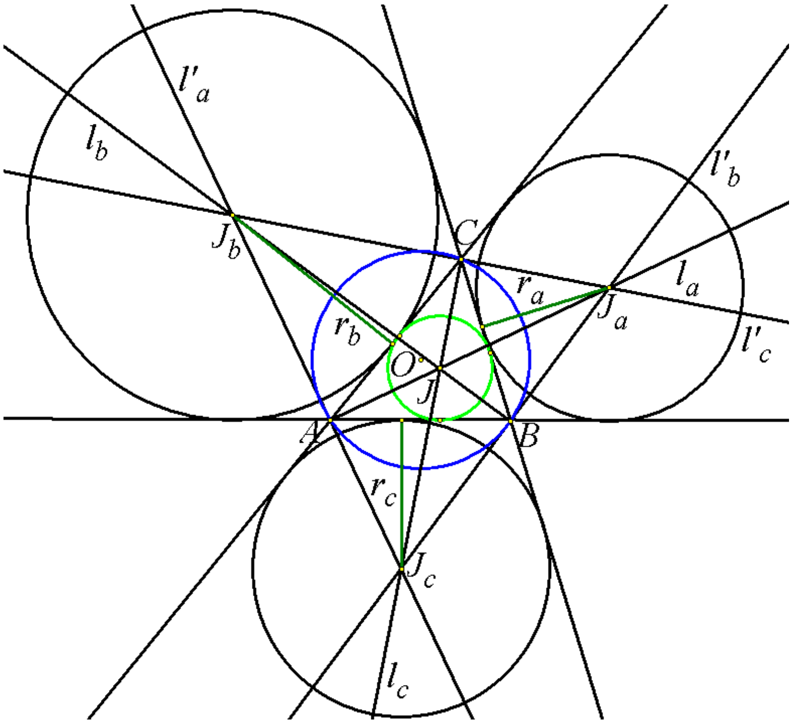

Fig. 4

If the internal angular bisectors through the vertices B and C are l_b and l_c, respectively, (Fig. 4), while the external angular bisectors through them are l'_b and l'_c, respectively, (Fig. 4), then using the equalities (5), (6), (2) and (4) we obtain:

$$\begin{aligned}(9)\qquad l_b:\ & \left\{(2r-1)(r-1)a^3+(4r-3)ja^2-(5r-3)a-j-\left[(2r-1)a^2+2ja-1\right]d\right\}z+\\ & +\left\{\left[ja^3+(5r-3)a^2-(4r-3)ja-(2r-1)(r-1)\right]a+\left(a^2-2ja-2r+1\right)d\right\}\overline{z}+\\ & +\left(a^2-1\right)\left[(2r-1)a^2-2\left(r^2+r-1\right)ja+2r-1\right]+\\ & +\left[2(r-1)ja^2-4(2r-1)a+2(r-1)j\right]d=0,\end{aligned}$$

$$\begin{aligned}(10)\qquad l'_b:\ & \left\{(2r-1)(r-1)a^3+(4r-3)ja^2-(5r-3)a-j-\left[(2r-1)a^2+2ja-1\right]d\right\}z-\\ & -\left\{\left[ja^3+(5r-3)a^2-(4r-3)ja-(2r-1)(r-1)\right]a+\left(a^2-2ja-2r+1\right)d\right\}\overline{z}-\\ & -(2r-1)a^4+2\left(r^2-2\right)ja^3-2\left(4r^2+2r-3\right)a^2-\\ & +2\left(r^2-2\right)ja-2r+1-2\left(a^2-1\right)rjd=0,\end{aligned}$$

$$\begin{aligned}(11)\qquad l_c:\ & \left\{(2r-1)(r-1)a^3+(4r-3)ja^2-(5r-3)a-j+\left[(2r-1)a^2+2ja-1\right]d\right\}z+\\ & +\left\{\left[ja^3+(5r-3)a^2-(4r-3)ja-(2r-1)(r-1)\right]a-\left(a^2-2ja-2r+1\right)d\right\}\overline{z}+\\ & +\left(a^2-1\right)\left[(2r-1)a^2-2\left(r^2+r-1\right)ja+2r-1\right]+\\ & -\left[2(r-1)ja^2-4(2r-1)a+2(r-1)j\right]d=0,\end{aligned}$$

(12) $$l_c':\ \begin{aligned}&\left\{(2r-1)(r-1)a^3+(4r-3)ja^2-(5r-3)a-j+\left[(2r-1)a^2+2ja-1\right]d\right\}z+\\&-\left\{\left[ja^3+(5r-3)a^2-(4r-3)ja-(2r-1)(r-1)\right]a-\left(a^2-2ja-2r+1\right)d\right\}\overline{z}+\\&-(2r-1)a^4+2\left(r^2-2\right)ja^3-2\left(4r^2+2r-3\right)a^2+\\&+2\left(r^2-2\right)ja-2r+1+2\left(a^2-1\right)rjd=0.\end{aligned}$$

4. Determination of the centers of the excircles. Let the centers of the excircles of ΔABC, which are tangent to BC, CA and AB, be J_a, J_b and J_c, respectively, (Fig. 4). Using the equalities (2) and (4), also the software product Maple, we find the affixes of those points in the following way:

Since $J_a=l_b'\cap l_c'$ (Fig. 4), the affix j_a of the point could be determined solving the system, which is composed of the equations (10) and (12). Thus, we obtain

(13) $$j_a=\frac{u_a}{v_a},\ \overline{j}_a=-\frac{\tilde{u}_a}{v_a},$$

where

$$\begin{aligned}u_a=&\left(4r^2-1\right)a^6-2\left(6r^2-r-3\right)ja^5-\left(24r^3-24r^2-20r+15\right)a^4+\\&+4\left(2r^3-4r^2-5r+5\right)ja^3-(2r-1)\left(4r^2+10r-15\right)a^2+2(2r-1)(r-3)ja-(2r-1)^2,\end{aligned}$$

$$\begin{aligned}v_a=&(2r-1)ja^6+6(2r-1)(r-1)a^5-3\left(4r^2-10r+5\right)ja^4-\\&-4(r-1)\left(2r^2-10r+5\right)a^3-3\left(4r^2-10r+5\right)ja^2+6(2r-1)(r-1)a+(2r-1)j,\end{aligned}$$

$$\begin{aligned}\tilde{u}_a=&(2r-1)^2a^6-2(2r-1)(r-3)ja^5+(2r-1)\left(4r^2+10r-15\right)a^4+\\&-4\left(2r^3-4r^2-5r+5\right)ja^3+\left(24r^3-24r^2-20r+15\right)a^2+2\left(6r^2-r-3\right)ja-4r^2+1.\end{aligned}$$

Since $J_b=l_b\cap l_c'$ (Fig. 4), the affix j_b of the point could be determined solving the system, which is composed of the equations (9) and (12). Thus, we obtain

(14) $$j_b=\frac{u_b}{v_b},\ \overline{j}_b=\frac{u_b}{\tilde{v}_b},$$

where

$$\begin{aligned}u_b=&-(2r-1)a^6+2(4r-3)ja^5+\left(24r^2-40r+15\right)a^4-4\left(4r^2-10r+5\right)ja^3+\\&-(2r-1)\left(4r^2-20r+15\right)a^2-2(2r-1)(2r-3)ja+(2r-1)^2+\\&+\left[2(2r-1)a^5-2(6r-5)ja^4-4\left(6r^2-12r+5\right)a^3+\right.\\&\left.+4\left(2r^2-8r+5\right)ja^2-2(2r-1)(4r-5)a-2(2r-1)j\right]d,\end{aligned}$$

$$\begin{aligned}v_b=&(2r-1)ja^6+6(2r-1)(r-1)a^5-3\left(4r^2-10r+5\right)ja^4-4(r-1)\left(2r^2-10r+5\right)a^3-\\&-3\left(4r^2-10r+5\right)ja^2+6(2r-1)(r-1)a+(2r-1)j,\end{aligned}$$

$$\begin{aligned}\tilde{u}_b=&\left[(2r-1)^2a^6-2(2r-1)(2r-3)ja^5-(2r-1)\left(4r^2-20r+15\right)a^4-4\left(4r^2-10r+5\right)ja^3+\right.\\&\left.+\left(24r^2-40r+15\right)a^2+2(4r-3)ja-2r+1\right]a-\\&+\left[2(2r-1)ja^5+2(2r-1)(4r-5)a^4-4\left(2r^2-8r+5\right)ja^3+\right.\\&\left.+4\left(6r^2-12r+5\right)a^2+2(6r-5)ja-2(2r-1)\right]d,\end{aligned}$$

$\tilde{v}_b = av_b$.

Since $J_c = l_b' \cap l_c$ (Fig. 4), the affix j_c of the point could be determined solving the system, which is composed of the equations (10) and (11). Thus, we obtain

$$(15) \qquad j_c = \frac{u_c}{v_c},\ \overline{j}_c = \frac{u_c}{\tilde{v}_c},$$

where

$$u_c = -(2r-1)a^6 + 2(4r-3)ja^5 + (24r^2-40r+15)a^4 - 4(4r^2-10r+5)ja^3 -$$
$$-(2r-1)(4r^2-20r+15)a^2 - 2(2r-1)(2r-3)ja + (2r-1)^2 -$$
$$-\left[2(2r-1)a^5 - 2(6r-5)ja^4 - 4(6r^2-12r+5)a^3 +\right.$$
$$\left.+4(2r^2-8r+5)ja^2 - 2(2r-1)(4r-5)a - 2(2r-1)j\right]d,$$

$$\tilde{u}_c = \left[(2r-1)^2a^6 - 2(2r-1)(2r-3)ja^5 - (2r-1)(4r^2-20r+15)a^4 - 4(4r^2-10r+5)ja^3 -\right.$$
$$\left.+(24r^2-40r+15)a^2 + 2(4r-3)ja + 2r - 1\right]a +$$
$$-\left[2(2r-1)a^5 + 2(2r-1)(4r-5)a^4 - 4(2r^2-8r+5)ja^3 +\right.$$
$$\left.+4(6r^2-12r+5)a^2 + 2(6r-5)ja - 2(2r-1)\right]d,$$

$$v_c = (2r-1)ja^6 + 6(2r-1)(r-1)a^5 - 3(4r^2-10r+5)ja^4 - 4(r-1)(2r^2-10r+5)a^3 -$$
$$-3(4r^2-10r+5)ja^2 + 6(2r-1)(r-1)a + (2r-1)j,$$

$\tilde{v}_c = av_c$.

5. Determination of the radii of the excircles. Euler formulae $OJ_a^2 = R^2 + 2Rr_a$, $OJ_b^2 = R^2 + 2Rr_b$ and $OJ_c^2 = R^2 + 2Rr_c$, concerning the distances from the center O of Γ to the centers J_a, J_b and J_c of the excircles, are used for the radii r_a, r_b and r_c (Fig. 4) to obtain the equalities:

$$(16) \qquad r_a = \frac{j_a\overline{j}_a - 1}{2},\ r_b = \frac{j_b\overline{j}_b - 1}{2},\ r_c = \frac{j_c\overline{j}_c - 1}{2}.$$

Substituting the equalities (13), (14) and (15) in (16), using the computing capabilities of Maple and the equalities (2) and (4), we obtain for the radii r_a, r_b and r_c, that:

$$(17) \qquad r_a = -\frac{r\xi_a}{\eta_a^2},\ r_b = \frac{\xi_b}{a\eta_a^2},\ r_c = \frac{\xi_c}{a\eta_a^2},$$

where

$$\xi_a = (2r-1)^3 a^{12} - 4(2r-1)^2 (2r-3) ja^{11} - 2(2r-1)^2 (10r^2 - 46r + 33) a^{10} -$$
$$-20(2r-1)(12r^2 - 24r + 11) ja^9 - 5(2r-1)(16r^4 + 32r^3 - 252r^2 + 300r - 99) a^8 +$$
$$+8(16r^5 - 40r^4 - 160r^3 + 480r^2 - 390r + 99) ja^7 +$$
$$+4(16r^6 - 144r^5 + 1120r^3 - 1890r^2 + 1134r - 231) a^6 +$$
$$+8(16r^5 - 40r^4 - 160r^3 + 480r^2 - 390r + 99) ja^5 -$$
$$-5(2r-1)(16r^4 + 32r^3 - 252r^2 + 300r - 99) a^4 - 20(2r-1)(12r^2 - 24r + 11) ja^3 -$$
$$-2(2r-1)^2 (10r^2 - 46r + 33) a^2 - 4(2r-1)^2 (2r-3) ja + (2r-1)^3,$$

$$\eta_a = (2r-1) ja^6 + 6(2r-1)(r-1) a^5 - 3(4r^2 - 10r + 5) ja^4 - 4(r-1)(2r^2 - 10r + 5) a^3 -$$
$$-3(4r^2 - 10r + 5) ja^2 + 6(2r-1)(r-1) a + (2r-1) j,$$

$\xi_b = -2ax_b - y_b d$, $\xi_c = -2ax_b + y_b d$,

$$x_b = (2r-1)^3 a^{12} - (2r-1)^2 (r^2 + 12r - 12) ja^{11} - 2(2r-1)^2 (5r^3 + 25r^2 - 66r + 33) a^{10} +$$
$$+5(2r-1)(8r^4 + 14r^3 - 111r^2 + 132r - 44) ja^9 +$$
$$+5(2r-1)(16r^5 - 16r^4 - 216r^3 + 516r^2 - 396r + 99) a^8 +$$
$$-2(40r^6 - 184r^5 - 400r^4 + 2460r^3 - 3495r^2 + 1980r - 396) ja^7 -$$
$$-4(8r^7 - 104r^6 + 84r^5 + 1120r^4 - 3045r^3 + 3087r^2 - 1386r + 231) a^6 -$$
$$-2(40r^6 - 184r^5 - 400r^4 + 2460r^3 - 3495r^2 + 1980r - 396) ja^5 +$$
$$+5(2r-1)(16r^5 - 16r^4 - 216r^3 + 516r^2 - 396r + 99) a^4 +$$
$$+5(2r-1)(8r^4 + 14r^3 - 111r^2 + 132r - 44) ja^3 -$$
$$-2(2r-1)^2 (5r^3 + 25r^2 - 66r + 33) a^2 - (2r-1)^2 (r^2 + 12r - 12) ja + (2r-1)^3,$$

$$y_b = -(2r-1)^3 a^{12} + 10(2r-1)^2 (r-1) ja^{11} + 4(2r-1)^2 (10r^2 - 22r + 11) a^{10} -$$
$$-10(2r-1)(r-1)(8r^2 - 22r + 11) ja^9 - 5(2r-1)(16r^4 - 96r^3 + 180r^2 - 132r + 33) a^8 +$$
$$+4(r-1)(8r^4 - 72r^3 + 168r^2 - 132r + 33) ja^7 -$$
$$-4(r-1)(8r^4 - 72r^3 + 168r^2 - 132r + 33) ja^5 +$$
$$+5(2r-1)(16r^4 - 96r^3 + 180r^2 - 132r + 33) a^4 + 10(2r-1)(r-1)(8r^2 - 22r + 11) ja^3 -$$
$$-4(2r-1)^2 (10r^2 - 22r + 11) a^2 - 10(2r-1)^2 (r-1) ja + (2r-1)^3.$$

6. Determination of the affix of Gergonne point. From the equality (1) we obtain for the affix $g = \dfrac{r_a a + r_b b + r_c c}{r_a + r_b + r_c}$ of Gergonne point. Substituting the equalities (5), (6) and (17), using the computing capabilities of Maple and the equalities (2) and (4), we obtain for the affix g of G, that:

(18) $$g = \frac{h_1}{(r+4)\left[(2r-1)a^2 + 2ja - 1\right] h_2}, \quad \overline{g} = \frac{h_1}{a.(r+4)(a^2 - 2ja - 2r + 1) h_2},$$

where

$h_1=(2r-1)^4 ra^{15}-2(2r-1)^3(4r-1)(r-2)ja^{14}-(2r-1)^3(16r^3-144r^2+195r-56)a^{13}-$
$-4(2r-1)^2(8r^4+84r^3-346r^2+338r-91)ja^{12}-$
$-(2r-1)^2(160r^5-192r^4-3424r^3+8288r^2-6201r+1456)a^{11}+$
$+2(2r-1)(64r^6-960r^5-480r^4+10032r^3-16500r^2+9867r-2002)ja^{10}-$
$-(2r-1)(256r^7+2304r^6-9632r^5-13120r^4+74800r^3-91696r^2+45331r-8008)a^9+$
$+4(2r-1)(r-1)(64r^6-1872r^5+528r^3+9108r^2-10296r+3003)ja^8+$
$+(2r-1)\times$
$\times(128r^8-1472r^7-2272r^6+23952r^5-10200r^4-81708r^3+129162r^2-71643r+13728)a^7+$
$+2(384r^8-1600r^7-6176r^6+23344r^5+920r^4-64724r^3+77990r^2-36179r+6006)ja^6-$
$-(2r-1)(960r^7-832r^6-14512r^5+22432r^4+21692r^3-60236r^2+38753r-8008)a^5-$
$-4(2r-1)(160r^6+240r^5-2112r^4+984r^3+3414r^2-3666r+1001)ja^4+$
$+(2r-1)^2(240r^5+928r^4-2536r^3-904r^2+3627r-1456)a^3+$
$+2(2r-1)^2(24r^4+172r^3-158r^2-241r+182)ja^2-4(2r-1)^3(4r^3+60r^2+9r-56)a-$
$-4(2r-1)^3(r+1)j$

$h_2=(2r-1)^3a^{12}-12(2r-1)^2(r-1)ja^{11}-6(2r-1)^2(10r^2-22r+11)a^{10}+$
$+20(2r-1)(r-1)(8r^2-22r+11)ja^9+$
$+15(2r-1)(16r^4-96r^3+180r^2-132r+33)a^8+$
$-24(r-1)(8r^4-72r^3+168r^2-132r+33)ja^7-$
$-4(16r^6-336r^5+1680r^4-3360r^3+3150r^2-1386r+231)a^6-$
$-24(r-1)(8r^4-72r^3+168r^2-132r+33)ja^5+$
$+15(2r-1)(16r^4-96r^3+180r^2-132r+33)a^4+$
$+20(2r-1)(r-1)(8r^2-22r+11)ja^3-$
$-6(2r-1)^2(10r^2-22r+11)a^2-12(2r-1)^2(r-1)ja+(2r-1)^3$

7. Equation of Poncelet-Gergonne circle. We look for equation of a circle $k(G):(z-p)(\overline{z}-p)=\rho^2$ with center P on the real axis OJ and radius ρ expecting to find the locus of Gergonne point. If $q=p^2-\rho^2$, then

(19) $$k(G): z\overline{z}-pz-p\overline{z}+q=0.$$

The circle $k(G)$ meets the real axis OJ in two different points G_1 and G_{-1} (Fig. 5). The affixes of those points are obtained from (18) when $a=1$ and $a=-1$, respectively (Fig. 5):

(20) $$g_1=\overline{g}_1=-\frac{(2r-1)\vartheta+\tau.j}{(r+4)(j-r+1)\varphi_1},\ g_{-1}=\overline{g}_{-1}=\frac{(2r-1)\vartheta-\tau.j}{(r+4)(j-r+1)\varphi_{-1}},$$

where

$\varphi_1=2(r-1)(r-2)(3r-2)(r^2-8r+4)j+(r^2-4r+2)(r^4-32r^3+80r^2-64r+16),$

$\varphi_{-1}=2(r-1)(r-2)(3r-2)(r^2-8r+4)j-(r^2-4r+2)(r^4-32r^3+80r^2-64r+16),$

$\tau = 10r^8 - 39r^7 - 252r^6 + 712r^5 + 704r^4 - 3440r^3 + 3648r^2 - 1600r + 256,$
$\vartheta = r^8 - 21r^7 - 28r^6 + 392r^5 - 144r^4 - 1552r^3 + 2432r^2 - 1344r + 256.$

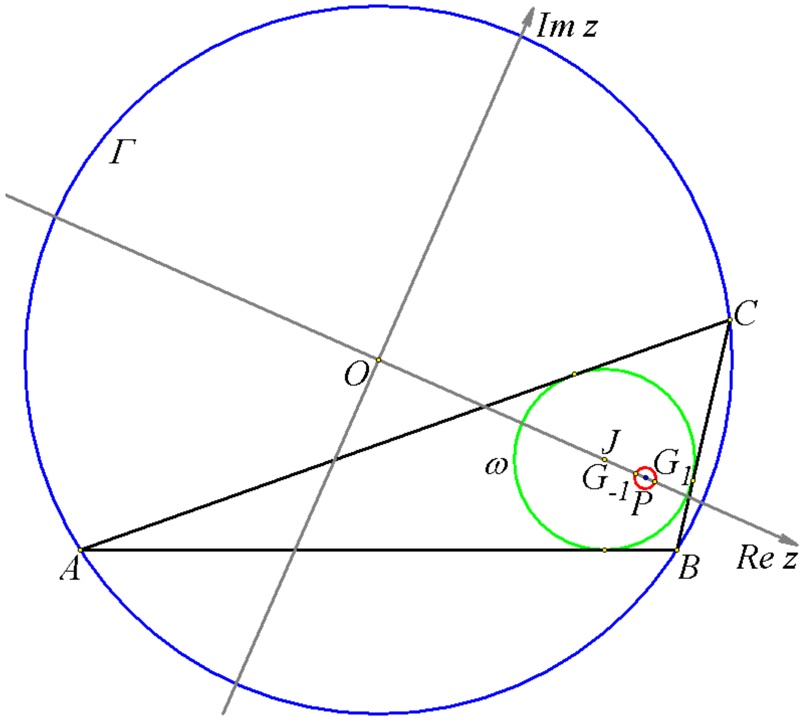

Fig. 5

After substituting (20) in (19) we obtain the next two equations with respect to the unknowns p and q:

$$2(r+4)\left[2x_p.j+(2r-1)y_p\right]p+(r+4)^2\left[2x_q.j+(r^2-4r+2)y_{q_1}\right]q=(1-2r)(2.\vartheta.\tau.j+z_0),$$
$$2(r+4)\left[2x_p.j-(2r-1)y_p\right]p-(r+4)^2\left[2x_q.j-(r^2-4r+2)y_q\right]q=(2r-1)(2.\vartheta.\tau.j-z_0),$$

where

$x_p = 12r^{15} - 527r^{14} + 5579r^{13} - 13988r^{12} - 70304r^{11} + 459680r^{10} -$
$-759968r^9 - 868224r^8 + 5749248r^7 - 11241216r^6 + 12618496r^5 -$
$-9069568r^4 + 4255744r^3 - 1265664r^2 + 217088r - 16384,$

$y_p = r^{15} - 140r^{14} + 2786r^{13} - 15288r^{12} - 7904r^{11} + 311168r^{10} -$
$919360r^9 + 248064r^8 + 4217088r^7 - 11303936r^6 + 15178240r^5 -$
$-12482560r^4 + 6549504r^3 - 2146304r^2 + 401408r - 32768,$

$x_q = (r-1)(7r^3 - 28r^2 + 28r - 8)(r^3 - 12r^2 + 20r - 8)\times$
$\times(r^6 - 48r^5 + 344r^4 - 832r^3 + 848r^2 - 384r + 64),$
$y_q = r^{12} - 192r^{11} + 5600r^{10} - 58752r^9 + 298336r^8 - 867328r^7 + 1577472r^6 -$
$-1878016r^5 + 1488128r^4 - 778240r^3 + 258048r^2 - 49152r + 4096,$

$z_0 = 2r^{17} - 185r^{16} + 1592r^{15} + 7054r^{14} - 67792r^{13} - 44096r^{12} + 965120r^{11} -$

$-986816r^{10} - 4839424r^9 + 12223232r^8 - 2275328r^7 - 29125120r^6 +$

$+53972992r^5 49352704r^4 + 26869760r^3 - 8855552r^2 + 1638400r - 131072.$

The last system of linear equations is solved by means of Maple

(21) $$p = \frac{4(1+r)j}{4+r},\ q = \frac{(1-2r)(4+7r+2r^2)}{4+r}.$$

From (21) and the equality $q = p^2 - \rho^2$, i.e. $\rho^2 = p^2 - q$, we obtain the radius ρ of the circle $k(G)$:

(22) $$\rho = \frac{(1-2r)r}{4+r}.$$

8. Checking the equation of Poncelet-Gergonne circle. The equation (19) of $k(G)$ is defined fully by the values of p and q according to the equalities (21). The circle itself is determined with two special points on it but this does not guarantee that each Gergonne point is on it when ΔABC moves. It is necessary for this reason to check whether the point G, defined with (18), is on the circle $k(G)$ in the general case. We substitute g and $\overline{g}$ from (18) in the left hand side of (19) and after several transformations with Maple we obtain

$$g\overline{g} - pg - p\overline{g} + q = \frac{(j^2+2r-1).\Delta}{(r+4)^2(a^2-2ja-2r+1)\left[(2r-1)a^2+2ja-1\right]g_2^2},$$

where

$\Delta = (2r-1)^6(r-1)(r-2)(4r-1)a^{28} - 2(2r-1)^5(r+1)(20r^3 - 102r^2 + 111r - 28)ja^{27} +$

$+2(2r-1)^5(12r^6 - 28r^5 + 801r^4 - 1194r^3 - 512r^2 + 1253r - 378)a^{26} -$

$-2(2r-1)^4(144r^7 + 496r^6 + 3364r^5 - 14224r^4 + 8134r^3 + 11923r^2 - 13087r + 3276)ja^{25} -$

$-(2r-1)^4(1760r^8 + 7136r^7 + 3480r^6 - 160368r^5 + 269828r^4 +$

$+92r^3 - 277141r^2 + 196805r - 40950)a^{24} +$

$+4(2r-1)^3(1920r^9 + 5456r^8 - 26312r^7 - 113196r^6 + 441934r^5 - 354926r^4 - 255441r^3 +$

$+530424r^2 - 279077r + 49140)ja^{23} +$

$+4(2r-1)^3(6336r^{10} + 4320r^9 - 146256r^8 - 40248r^7 + 1589884r^6 - 2865344r^5 +$

$+728537r^4 + 2635418r^3 - 2965980r^2 + 1241025r - 188370)a^{22} -$

$-4(2r-1)^2(15104r^{11} - 29696r^{10} - 403552r^9 + 919264r^8 + 2958256r^7 - 11803176r^6 +$

$+11618978r^5 + 3921764r^4 - 16148421r^3 + 12812134r^2 - 4452389r + 592020)ja^{21} -$

$-(2r-1)^2(101376r^{12} - 544768r^{11} - 2411136r^{10} + 15076608r^9 +$

$+279744r^8 - 113771456r^7 + 225242472r^6 - 103554160r^5 -$

$-175473232r^4 + 285767244r^3 - 177137323r^2 + 52648607r - 6216210)a^{20} +$

$+2(2r-1)(61440r^{13} - 624640r^{12} - 430080r^{11} + 15755136r^{10} - 30391872r^9 +$

$-69115712r^8 + 310060832r^7 - 363409368r^6 - 18433756r^5 +$

$+454317712r^4 - 490425388r^3 + 250863943r^2 - 65136577r + 6906900)ja^{19} +$

$+2(2r-1)(56320r^{14} - 939008r^{13} + 2214144r^{12} + 18679808r^{11} - 87366080r^{10} +$

$+25002560r^9 + 506791376r^8 - 1141561856r^7 + 754495772r^6 + 675594692r^5 -$

$-1640106945r^4 + 1355949586r^3 - 592192488r^2 + 136497243r - 13123110)a^{18} -$

$$-2(2r-1)\big(18432r^{14}-472064r^{13}+2493440r^{12}+5320192r^{11}-63552768r^{10}+$$
$$+100728576r^{9}+236019264r^{8}-953223456r^{7}+1021964064r^{6}+198308376r^{5}-$$
$$-1487940772r^{4}+1536940678r^{3}-776550637r^{2}+201423429r-21474180\big)ja^{17}+$$
$$-(2r-1)\big(12288r^{15}-636928r^{14}+6313984r^{13}-8215552r^{12}-117096960r^{11}+$$
$$+455728896r^{10}-99222528r^{9}-2398756096r^{8}+5109230720r^{7}-$$
$$-3015239232r^{6}-3590341920r^{5}+7801734640r^{4}-6310928918r^{3}+$$
$$+2737247435r^{2}-630648985r+60843510\big)a^{16}+$$
$$-8\big(31744r^{15}-589824r^{14}+2668800r^{13}+5489664r^{12}-65107200r^{11}+120740864r^{10}+$$
$$+169336384r^{9}-930060288r^{8}+1251532648r^{7}-182956416r^{6}-1450565514r^{5}+$$
$$+2012372872r^{4}-1349049137r^{3}+513098904r^{2}-106303183r+9360540\big)ja^{15}-$$
$$-8\big(6144r^{16}-263168r^{15}+2432000r^{14}-3536640r^{13}-40846464r^{12}+$$
$$+176326272r^{11}-107689472r^{10}-780932416r^{9}+2060678704r^{8}-$$
$$-1770760152r^{7}-765449864r^{6}+3212188334r^{5}-3380975613r^{4}+$$
$$+1943436050r^{3}-658478198r^{2}+123897131r-10029150\big)a^{14}+$$
$$-8\big(31744r^{15}-589824r^{14}+2668800r^{13}+5489664r^{12}-65107200r^{11}+$$
$$+120740864r^{10}+169336384r^{9}-930060288r^{8}+1251532648r^{7}-$$
$$-182956416r^{6}-1450565514r^{5}+2012372872r^{4}-1349049137r^{3}+$$
$$+513098904r^{2}-106303183r+9360540\big)ja^{13}+$$
$$-(2r-1)\big(12288r^{15}-636928r^{14}+6313984r^{13}-8215552r^{12}-$$
$$-117096960r^{11}+455728896r^{10}-99222528r^{9}-2398756096r^{8}+$$
$$+5109230720r^{7}-3015239232r^{6}-3590341920r^{5}+7801734640r^{4}-$$
$$-6310928918r^{3}+2737247435r^{2}-630648985r+60843510\big)a^{12}-$$
$$-2(2r-1)\big(18432r^{14}-472064r^{13}+2493440r^{12}+5320192r^{11}-$$
$$-63552768r^{10}+100728576r^{9}+236019264r^{8}-953223456r^{7}+$$
$$+1021964064r^{6}+198308376r^{5}-1487940772r^{4}+1536940678r^{3}-$$
$$-776550637r^{2}+201423429r-21474180\big)ja^{11}+$$
$$+2(2r-1)\big(56320r^{14}-939008r^{13}+2214144r^{12}+18679808r^{11}-$$
$$-87366080r^{10}+25002560r^{9}+506791376r^{8}-1141561856r^{7}+$$
$$+754495772r^{6}+675594692r^{5}-1640106945r^{4}+$$
$$+1355949586r^{3}-592192488r^{2}+136497243r-13123110\big)a^{10}+$$
$$+2(2r-1)\big(61440r^{13}-624640r^{12}-430080r^{11}+15755136r^{10}-$$
$$-30391872r^{9}-69115712r^{8}+310060832r^{7}-363409368r^{6}-18433756r^{5}+$$
$$+\big(454317712r^{4}-490425388r^{3}+250863943r^{2}-65136577r+6906900\big)ja^{9}-$$
$$-(2r-1)^{2}\big(101376r^{12}-544768r^{11}-2411136r^{10}+15076608r^{9}+279744r^{8}+$$
$$-113771456r^{7}+225242472r^{6}-103554160r^{5}-175473232r^{4}+$$
$$+285767244r^{3}-177137323r^{2}+52648607r-6216210\big)a^{8}-$$
$$-4(2r-1)^{2}\big(15104r^{11}-29696r^{10}-403552r^{9}+919264r^{8}+$$
$$+2958256r^{7}-11803176r^{6}+11618978r^{5}+3921764r^{4}-$$
$$-16148421r^{3}+12812134r^{2}-4452389r+592020\big)ja^{7}+$$
$$+4(2r-1)^{3}\big(6336r^{10}+4320r^{9}-146256r^{8}-40248r^{7}+1589884r^{6}-$$

$-2865344r^5+728537r^4+2635418r^3-2965980r^2+1241025r-188370)a^6+$
$+4(2r-1)^3(1920r^9+5456r^8-26312r^7-113196r^6+441934r^5-$
$-354926r^4-255441r^3+530424r^2-279077r+49140)ja^5-$
$-(2r-1)^4(1760r^8+7136r^7+3480r^6-160368r^5+$
$+269828r^4+92r^3-277141r^2+196805r-40950)a^4-$
$-2(2r-1)^4(144r^7+496r^6+3364r^5-14224r^4+8134r^3+11923r^2-13087r+3276)ja^3+$
$+2(2r-1)^5(12r^6-28r^5+801r^4-1194r^3-512r^2+1253r-378)a^2+$
$-2(2r-1)^5(r+1)(20r^3-102r^2+111r-28)a+1472r^7$

Since $j^2+2r-1=0$ according to (2), the equation $g\overline{g}-pg-p\overline{g}+q=0$ is satisfied for each point G. Therefore, Gergonne point G describes the established circle $k(G)$, when ΔABC moves between the circles Γ and ω.

9. Conclusion. The results (21) and (22) are obtained with calculations under the assumption $R=1$. Accounting for that fact and the equality $|j|=OJ$, we note that in the general case the equalities (21) and (22) induce the following formulae:

$$OP=\frac{4(R+r).OJ}{4R+r}, \tag{23}$$

$$\rho=\frac{(R-2r)r}{4R+r}. \tag{24}$$

Accounting for the equalities (23) and (24), we could clarify assertion 1 by the following

Theorem. *If G is Gergonne point for the moving triangle ABC between the circles Γ and ω, then it describes the circle $k(G)$ with center P on the line OJ in such a way, that* $OP=\frac{4(R+r).OJ}{4R+r}$ *and the radius is* $\rho=\frac{(R-2r)r}{4R+r}$ (Fig. 3, 5).

In addition, it is noticed from the formulae (23) and (24) that the locus which describes the point G, coincides with the point O iff the circles Γ and ω are concentric resulting an equilateral triangle *ABC*.

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