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Exclusion of Trivial Angle Relationships in the Analysis of Geometrical Figures

Stanley Rabinowitz[a] and Ercole Suppa[b]
[a] 545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net
web: http://www.StanleyRabinowitz.com
[b] Via B. Croce 54, 64100 Teramo, Italia
e-mail: ercolesuppa@gmail.com
web: https://www.esuppa.it

Abstract. Given the coordinates for a set of points and a sequence of constructions, it is easy for a computer program to find all the pairs of equal angles in the figure formed by applying the constructions to these points. However, many of these results are trivial or follow from well-known theorems about angles. This paper describes how the program GeometricExplorer detects and excludes such trivial results. We also present a number of new results discovered by this program.

Keywords. GeometricExplorer, Mathematica, computer-discovered mathematics.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

Given the coordinates for a set of points and a sequence of constructions, it is easy for a computer program to find all the pairs of equal angles in the figure formed by applying the constructions to these points.

For example, Figure 1 shows a diagram suggested by a problem by Yura Biletsky [3]. It consists of a square $ABCD$ with four construction steps: E is the midpoint of AB, F is the midpoint of AD, G is the midpoint of BE, and H is the intersection of BF and DG. There are 217 pairs of angles that are equal in this figure. This includes angles formed by any three points in the figure, not just between lines appearing in the figure. The list of equal angles is shown below.

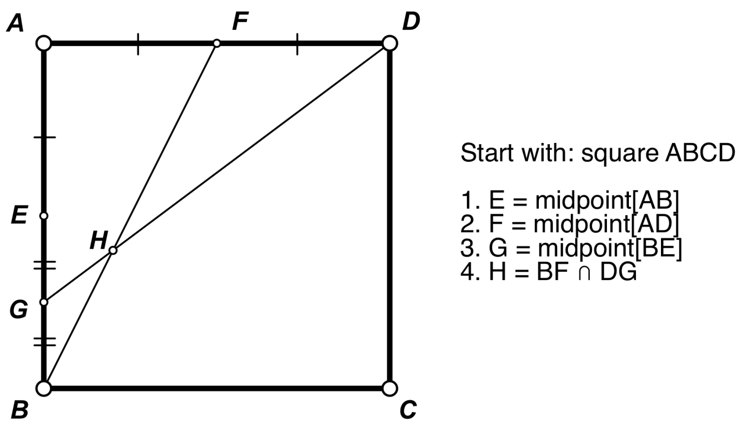


FIGURE 1. A square with Four Constructions

$\angle ACB=\angle ADB$, $\angle ACB=\angle AFE$, $\angle ACB=\angle AHE$, $\angle ACB=\angle BAC$, $\angle ACB=\angle BDC$, $\angle ACB=\angle CAD$,
$\angle ACB=\angle CBD$, $\angle ACB=\angle DBA$, $\angle ACB=\angle DCA$, $\angle ACB=\angle FEA$, $\angle ACB=\angle FHA$, $\angle ACE=\angle BAH$,
$\angle ACE=\angle DBF$, $\angle ACE=\angle DEF$, $\angle ACE=\angle EDB$, $\angle ACE=\angle EFB$, $\angle ACE=\angle FCA$, $\angle ADB=\angle AFE$,
$\angle ADB=\angle AHE$, $\angle ADB=\angle BAC$, $\angle ADB=\angle BDC$, $\angle ADB=\angle CAD$, $\angle ADB=\angle CBD$, $\angle ADB=\angle DBA$,
$\angle ADB=\angle DCA$, $\angle ADB=\angle FEA$, $\angle ADB=\angle FHA$, $\angle ADC=\angle BAD$, $\angle ADC=\angle BHC$, $\angle ADC=\angle CBA$,
$\angle ADC=\angle CHF$, $\angle ADC=\angle DCB$, $\angle ADC=\angle EHB$, $\angle ADC=\angle FHE$, $\angle ADE=\angle DCF$, $\angle ADE=\angle DHF$,
$\angle ADE=\angle ECB$, $\angle ADE=\angle FBA$, $\angle ADE=\angle GHB$, $\angle ADE=\angle HAC$, $\angle ADG=\angle FCE$, $\angle AFB=\angle BEC$,
$\angle AFB=\angle CBF$, $\angle AFB=\angle CFD$, $\angle AFB=\angle CHD$, $\angle AFB=\angle DCE$, $\angle AFB=\angle DEA$, $\angle AFB=\angle EDC$,
$\angle AFB=\angle EHG$, $\angle AFB=\angle FCB$, $\angle AFC=\angle BED$, $\angle AFC=\angle BFD$, $\angle AFC=\angle CEA$, $\angle AFC=\angle DHE$,
$\angle AFC=\angle GHC$, $\angle AFE=\angle AHE$, $\angle AFE=\angle BAC$, $\angle AFE=\angle BDC$, $\angle AFE=\angle CAD$, $\angle AFE=\angle CBD$,
$\angle AFE=\angle DBA$, $\angle AFE=\angle DCA$, $\angle AFE=\angle FEA$, $\angle AFE=\angle FHA$, $\angle AHB=\angle BEF$, $\angle AHB=\angle CHA$,
$\angle AHB=\angle EFD$, $\angle AHE=\angle BAC$, $\angle AHE=\angle BDC$, $\angle AHE=\angle CAD$, $\angle AHE=\angle CBD$, $\angle AHE=\angle DBA$,
$\angle AHE=\angle DCA$, $\angle AHE=\angle FEA$, $\angle AHE=\angle FHA$, $\angle BAC=\angle BDC$, $\angle BAC=\angle CAD$, $\angle BAC=\angle CBD$,
$\angle BAC=\angle DBA$, $\angle BAC=\angle DCA$, $\angle BAC=\angle FEA$, $\angle BAC=\angle FHA$, $\angle BAD=\angle BHC$, $\angle BAD=\angle CBA$,
$\angle BAD=\angle CHF$, $\angle BAD=\angle DCB$, $\angle BAD=\angle EHB$, $\angle BAD=\angle FHE$, $\angle BAH=\angle DBF$, $\angle BAH=\angle DEF$,
$\angle BAH=\angle EDB$, $\angle BAH=\angle EFB$, $\angle BAH=\angle FCA$, $\angle BDC=\angle CAD$, $\angle BDC=\angle CBD$, $\angle BDC=\angle DBA$,
$\angle BDC=\angle DCA$, $\angle BDC=\angle FEA$, $\angle BDC=\angle FHA$, $\angle BEC=\angle CBF$, $\angle BEC=\angle CFD$, $\angle BEC=\angle CHD$,
$\angle BEC=\angle DCE$, $\angle BEC=\angle DEA$, $\angle BEC=\angle EDC$, $\angle BEC=\angle EHG$, $\angle BEC=\angle FCB$, $\angle BED=\angle BFD$,
$\angle BED=\angle CEA$, $\angle BED=\angle DHE$, $\angle BED=\angle GHC$, $\angle BEF=\angle CHA$, $\angle BEF=\angle EFD$, $\angle BFC=\angle CED$,
$\angle BFC=\angle DGA$, $\angle BFC=\angle GDC$, $\angle BFD=\angle CEA$, $\angle BFD=\angle DHE$, $\angle BFD=\angle GHC$, $\angle BGC=\angle DCG$,
$\angle BHC=\angle CBA$, $\angle BHC=\angle CHF$, $\angle BHC=\angle DCB$, $\angle BHC=\angle EHB$, $\angle BHC=\angle FHE$, $\angle BHD=\angle FHG$,
$\angle CAD=\angle CBD$, $\angle CAD=\angle DBA$, $\angle CAD=\angle DCA$, $\angle CAD=\angle FEA$, $\angle CAD=\angle FHA$, $\angle CBA=\angle CHF$,
$\angle CBA=\angle DCB$, $\angle CBA=\angle EHB$, $\angle CBA=\angle FHE$, $\angle CBD=\angle DBA$, $\angle CBD=\angle DCA$, $\angle CBD=\angle FEA$,
$\angle CBD=\angle FHA$, $\angle CBF=\angle CFD$, $\angle CBF=\angle CHD$, $\angle CBF=\angle DCE$, $\angle CBF=\angle DEA$, $\angle CBF=\angle EDC$,
$\angle CBF=\angle EHG$, $\angle CBF=\angle FCB$, $\angle CEA=\angle DHE$, $\angle CEA=\angle GHC$, $\angle CED=\angle DGA$, $\angle CED=\angle GDC$,
$\angle CEF=\angle DHA$, $\angle CEF=\angle EFC$, $\angle CEF=\angle HAD$, $\angle CFD=\angle CHD$, $\angle CFD=\angle DCE$, $\angle CFD=\angle DEA$,
$\angle CFD=\angle EDC$, $\angle CFD=\angle EHG$, $\angle CFD=\angle FCB$, $\angle CHA=\angle EFD$, $\angle CHD=\angle DCE$, $\angle CHD=\angle DEA$,
$\angle CHD=\angle EDC$, $\angle CHD=\angle EHG$, $\angle CHD=\angle FCB$, $\angle CHF=\angle DCB$, $\angle CHF=\angle EHB$, $\angle CHF=\angle FHE$,
$\angle DBA=\angle DCA$, $\angle DBA=\angle FEA$, $\angle DBA=\angle FHA$, $\angle DBF=\angle DEF$, $\angle DBF=\angle EDB$, $\angle DBF=\angle EFB$,
$\angle DBF=\angle FCA$, $\angle DCA=\angle FEA$, $\angle DCA=\angle FHA$, $\angle DCB=\angle EHB$, $\angle DCB=\angle FHE$, $\angle DCE=\angle DEA$,
$\angle DCE=\angle EDC$, $\angle DCE=\angle EHG$, $\angle DCE=\angle FCB$, $\angle DCF=\angle DHF$, $\angle DCF=\angle ECB$, $\angle DCF=\angle FBA$,
$\angle DCF=\angle GHB$, $\angle DCF=\angle HAC$, $\angle DEA=\angle EDC$, $\angle DEA=\angle EHG$, $\angle DEA=\angle FCB$, $\angle DEF=\angle EDB$,
$\angle DEF=\angle EFB$, $\angle DEF=\angle FCA$, $\angle DGA=\angle GDC$, $\angle DHA=\angle EFC$, $\angle DHA=\angle HAD$, $\angle DHE=\angle GHC$,
$\angle DHF=\angle ECB$, $\angle DHF=\angle FBA$, $\angle DHF=\angle GHB$, $\angle DHF=\angle HAC$, $\angle ECB=\angle FBA$, $\angle ECB=\angle GHB$,
$\angle ECB=\angle HAC$, $\angle EDB=\angle EFB$, $\angle EDB=\angle FCA$, $\angle EDC=\angle EHG$, $\angle EDC=\angle FCB$, $\angle EFB=\angle FCA$,
$\angle EFC=\angle HAD$, $\angle EHB=\angle FHE$, $\angle EHG=\angle FCB$, $\angle FBA=\angle GHB$, $\angle FBA=\angle HAC$, $\angle FEA=\angle FHA$,
$\angle GHB=\angle HAC$

This list was found by the computer program, GeometricExplorer, which was designed to look for interesting relationships in geometric figures. Unfortunately, many of the relationships found are not very interesting. For example, the first relationship, $\angle ACB = \angle ADB$ is trivial. Each angle clearly has measure 45°.

With so many relationships, it is hard to tell if any of them are interesting "discoveries". By a *discovery*, we mean a relationship that is not obvious, trivial, well-known, or easy to prove. A report listing all nontrivial relationships found in a given figure is called a *discovery report*.

In this paper, we describe the algorithms used by GeometricExplorer to weed out the trivial angle relationships it finds, so that it will be easier for a user to locate the interesting discoveries. This information may be useful to other developers who are writing similar programs.

2. Equal Angles

One of the many properties that GeometricExplorer checks for is the property that two angles in a figure are equal. By *equal*, we mean that the two angles have the same measure, or equivalently, that the two angles are congruent.

2.1. Identical Angles.

Figure 2 shows an angle. An angle consists of a vertex (in this case point B) and the two rays comprising the sides of the angle (in this case, rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$). The vertex of an angle must be a known, labeled point in the figure. If two lines in the figure cross, but the crossing point is not a named point in the figure, then GeometricExplorer will not consider the angle at the crossing point when searching for properties of the figure.

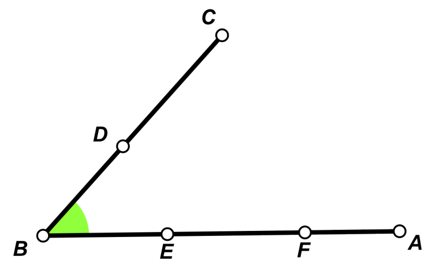


Figure 2. Many ways to name an angle

The angle shown can have many names. We want to exclude relationships like $\angle ABC = \angle CBA$ and $\angle FBC = \angle EBD$ from the report of relationships found. These relationships are trivial because the angles referenced are all identical.

In order to exclude identical angles from reports, we assign a *canonical name* to each angle in the figure. The canonical name for an angle is of the form $\angle XYZ$ where

(1) point Y is the vertex of the angle,
(2) the path $X \to Y \to Z$ proceeds in a clockwise direction,
(3) of all the known points that lie along ray $\overrightarrow{YX}$ (other than Y), X is the point whose name appears earliest in the alphabet, and
(4) of all the known points that lie along ray $\overrightarrow{YZ}$ (other than Y), Z is the point whose name appears earliest in the alphabet.

By this naming scheme, the angle shown in Figure 2 is named $\angle ABC$.

GeometricExplorer only considers angles whose measure lies strictly between 0° and 180°.

Using this naming scheme, two angles are identical if and only if they have the same canonical name. By always using canonical angle names, it is not necessary to perform tests to check if two angles are identical.

2.2. **Base Angles of Isosceles Triangles.** One of the most well-known theorems in geometry is that the base angles of an isosceles triangle are equal (Euclid I.5). If a figure contains a triangle ABC and $AB = AC$, then the fact that $\angle CBA = \angle ACB$ should clearly be considered a trivial result (Figure 3).

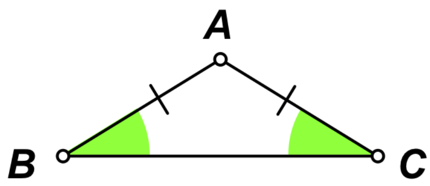


FIGURE 3. Base angles of an isosceles triangle

So how should GeometricExplorer check a discovered angle equality to see if it is trivial because the angles are base angles of an isosceles triangle? It might seem that $\angle CBA = \angle DEF$ should declared trivial if B coincides with F, A coincides with D, C coincides with E, and $AB = AC$. But it is not that simple because of the way angles are named. Although an angle in an isosceles triangle is named $\angle CBA$, this does not mean that AB is a side of the triangle.

The following is the algorithm that is actually used.

Algorithm B. Detect Base Angles of an Isosceles Triangle

If

(1) `aEqual=True`
(2) `exclude$isoscelesTriangles=True`
(3) $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$
(4) line P_1P_2 coincides with line Q_2Q_3
(5) $P_2P_3 \cap Q_1Q_2 = X$ and X is a known point in the figure
(6) $P_2X = Q_2X$

then exclude the result $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$ as trivial (Figure 4).

Note that this algorithm also finds other pairs of equal angles associated with an isosceles triangle (the pairs of green angles shown in Figure 4).

GeometricExplorer uses the *control variable* `aEqual` to determine whether or not to check for equal angles in a generated figure. If `aEqual` is set,the program will check for equal angles. GeometricExplorer uses the *exclusion variable* `exclude$isoscelesTriangles` to specify whether or not algorithm B should be used to exclude equal angles that are trivial because they are base angles of an isosceles triangle. If `exclude$isoscelesTriangles` is set, then such equal angles are excluded from reports about equal angles.

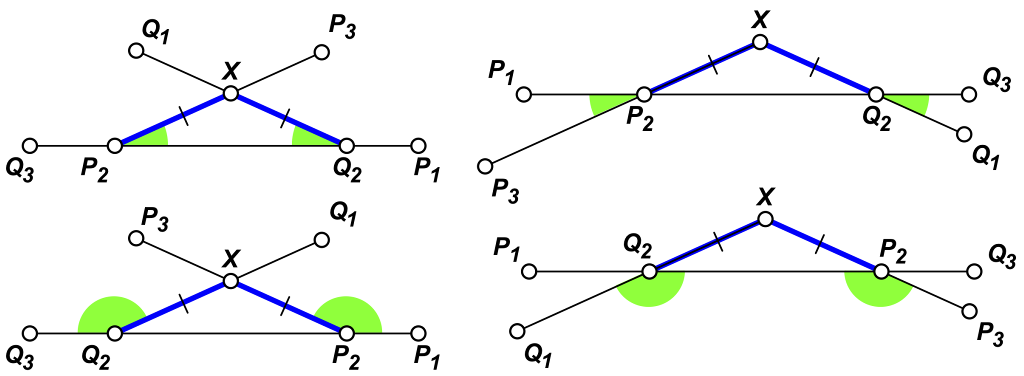


Figure 4. Detecting base angles of isosceles triangles

In order to check that existing code hasn't broken when new features are added, GeometricExplorer runs a series of *regression tests* to check that implemented features work correctly. If the `exclude$isoscelesTriangles` exclusion is working properly, then no nontrivial equal angles should be reported for the following figure. However, if `exclude$isoscelesTriangles` is turned off, then the result $\angle ADB = \angle BAC$ should be discovered.

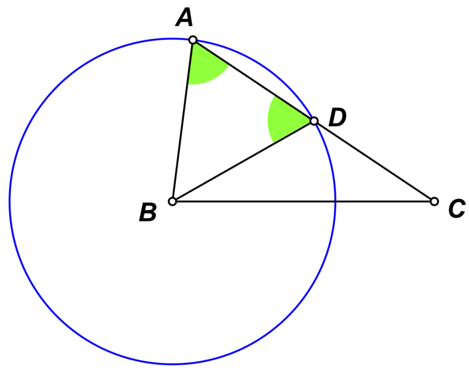


Figure 5. Regression Test for `exclude$isoscelesTriangles`

Regression Test for `exclude$isoscelesTriangles`

Start with: $\triangle ABC$ with $AC > AB$.
(1) Construct circle $B(A)$ with center B and passing through A.
(2) Let D be the intersection of AC and circle $B(A)$.
Conclusion: $\angle ADB = \angle BAC$

2.3. **Transversal to Parallel Lines.** It is well known (Euclid I.15 and I.29) that if two lines cross or a transversal meets two parallel lines (Figure 6), then

(1) vertical angles are equal ($\angle 1 = \angle 3$, $\angle 2 = \angle 4$, $\angle 5 = \angle 8$, $\angle 6 = \angle 7$)
(2) corresponding angles are equal ($\angle 1 = \angle 5$, $\angle 4 = \angle 7$, $\angle 2 = \angle 6$, $\angle 3 = \angle 8$)
(3) alternate interior angles are equal ($\angle 4 = \angle 6$, $\angle 3 = \angle 5$)
(4) alternate exterior angles are equal ($\angle 1 = \angle 8$, $\angle 2 = \angle 7$)

These equal angle results are all considered trivial and are excluded by the following algorithm.

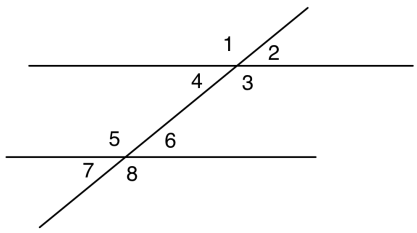


FIGURE 6. Angles formed by a transversal to two parallel lines

Algorithm P.
Detect Equal Angles formed by Angles Whose Sides are Parallel
If

(1) `aEqual=True`
(2) `exclude$parallelLines=True`
(3) $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$
(4) $P_1P_2 \parallel Q_1Q_2$ and $P_2P_3 \parallel Q_2Q_3$

then exclude the result $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$ as trivial (Figure 7).

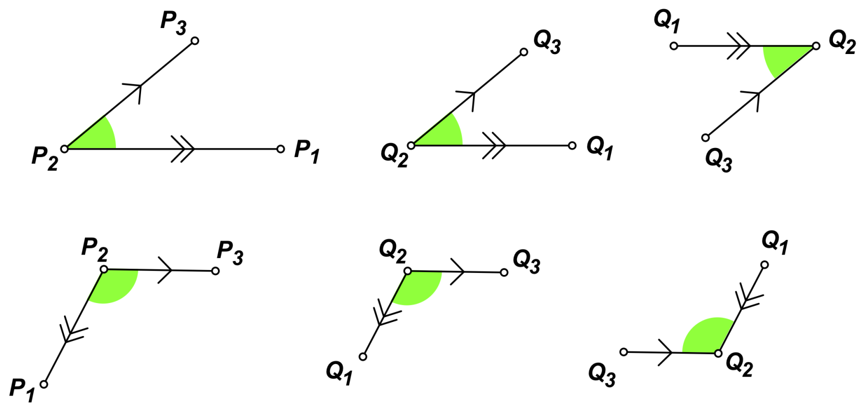


FIGURE 7. Detecting equal angles formed by two parallel rays

Note that algorithm P covers all the equal angles shown in Figure 6 as well as other pairs of equal angles whose sides consist of parallel rays, such as the ones shown in Figure 8.

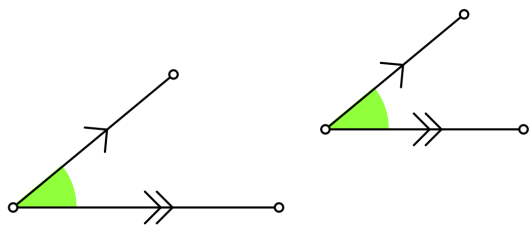


FIGURE 8. Angles formed by two parallel rays

2.4. **Overlapping Angles.** If two equal angles overlap and share a vertex, they can be excluded from reports. For example, in Figure 9, equal angles $\angle ABE$ and $\angle DBC$ (marked in blue) share vertex B and overlap. Subtracting from $\angle ABC$, we see that the yellow angles ($\angle ABD$ and $\angle EBC$) must also be equal. Since the equal nonoverlapping yellow angles will be included in the report listing equal angles, there is no need to also list the trivial consequence, $\angle ABE = \angle DBC$.

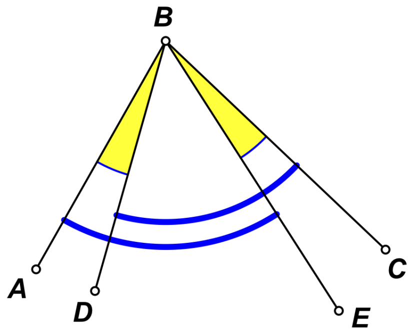


Figure 9. Overlapping angles

This exclusion is detected via the following algorithm.

Algorithm O. Detect Overlapping Angles

If

(1) `aEqual=True`
(2) `exclude$overlappingAngles=True`
(3) $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$
(4) $P_2 = Q_2$
(5) $\angle P_1P_2Q_1 + \angle Q_1P_2P_3 = \angle P_1P_2P_3$

then exclude the result $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$ as trivial (Figure 10).

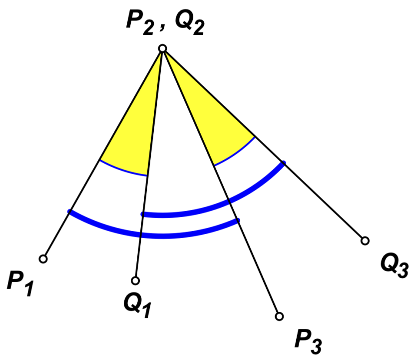


Figure 10. Detecting Overlapping angles

2.5. **Angles Associated with Cyclic Quadrilaterals.** There are many pairs of equal angles associated with a cyclic quadrilateral. Angles subtended in the same arc are equal, so $\angle DBA = \angle DCA$ in Figure 11 (left). In fact, all the green angles are equal.

An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle. So $\angle CDE = \angle CBA$ in Figure 11 (right). In fact, all the green angles are equal.

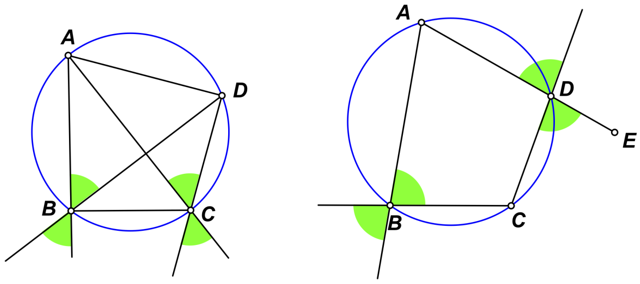


FIGURE 11. Equal Angles in Cyclic Quadrilaterals

Algorithm Q. Detect Equal Angles in a Cyclic Quadrilateral
If

(1) `aEqual=True`
(2) `exclude$cyclicQuadrilaterals=True`
(3) $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$
(4) $P_1P_2 \cap Q_1Q_2 = Y$ and Y is a known point in the figure
(5) $P_2P_3 \cap Q_2Q_3 = X$ and X is a known point in the figure
(6) $P_2 \neq Q_2$
(7) $P_3 \neq Q_1$
(8) $P_1 \neq Q_3$
(9) P_2, Q_2, X, Y are concylic

then exclude the result $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$ as trivial (Figure 12).

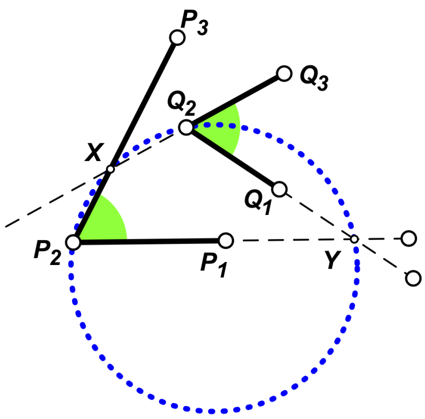


FIGURE 12. Detecting Equal Angles in Cyclic Quadrilaterals

A second test is also needed to handle a different configuration.

Algorithm Q2. Detect Equal Angles Associated with a Cyclic Quadrilateral
If

(1) `aEqual=True`
(2) `exclude$cyclicQuadrilaterals=True`

(3) $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$
(4) $P_1P_2 \cap Q_1Q_2 = X$ and X is a known point in the figure
(5) $P_2P_3 \cap Q_2Q_3 = Y$ and Y is a known point in the figure
(6) $P_2 \neq Q_2$
(7) P_2, Q_2, X, Y are concylic

then exclude the result $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$ as trivial (Figure 13).

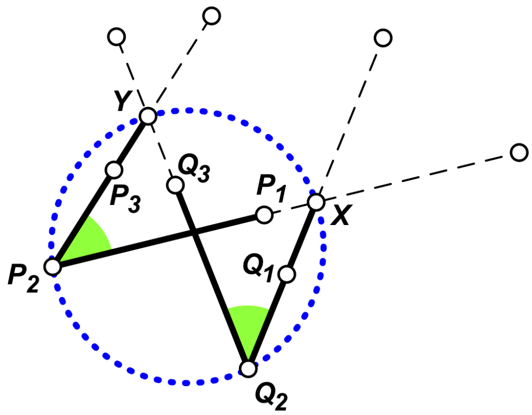


Figure 13. Detecting Equal Angles in Cyclic Quadrilaterals

2.6. **Angles Formed by a Chord and Tangent.** It is well known that if a chord of a circle has one endpoint at the point of tangency of a tangent to that circle, then the angle between the chord and the tangent is measured by half the measure of the intercepted arc. Thus, for example, in Figure 14, the two green angles are equal because both intercept the same arc. The equality of such angles is considered well known and should not be included in a report listing equal angles in a figure.

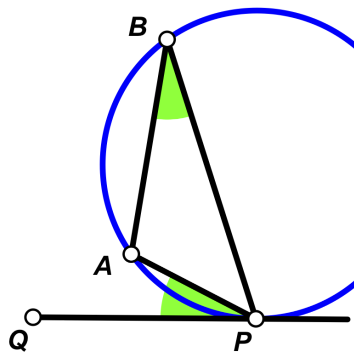


Figure 14. Equal angles formed by a tangent and a chord

Algorithm T. Detect Equal Angles Formed by a Tangent and a Chord
If

(1) `aEqual=True`
(2) `exclude$tangents=True`
(3) $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$

(4) P_2, Q_2, P_3 are collinear
(5) $P_1P_2 \cap Q_1Q_2 = X$ and X is a known point in the figure

then exclude the result $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$ as trivial (Figure 15).

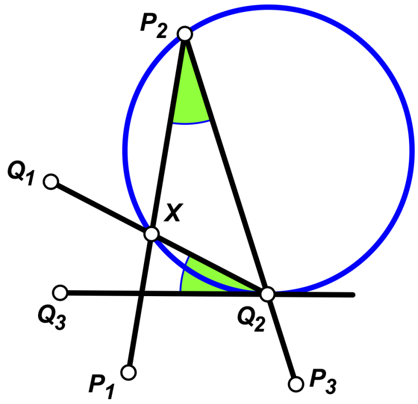


FIGURE 15. Detecting Equal Angles Formed by a Tangent and a Chord

Here is an example of an angle equality that is excluded because of Algorithm T. In Figure 16, we started with a triangle ABC. We then constructed a point D inside the triangle. Then we used an LPP construction to construct a circle tangent to line BC and passing through points A and D. The point of tangency of the circle with BC is named E. The conclusion $\angle DEB = \angle DAE$ is excluded from discovery reports because of Algorithm T.

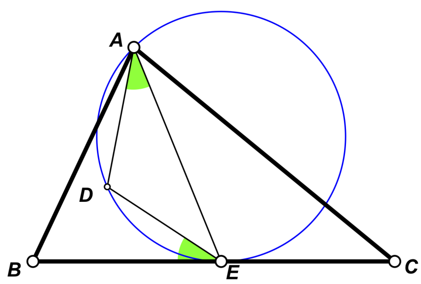


FIGURE 16. Equal Angles Formed by a Tangent and a Chord

2.7. **Angles Subended by Equal Chords.** Angles subtended by equal chords in congruent circles are well known to be equal. So such equal angles should not be reported.

Algorithm H. Detect Equal Angles Formed by Equal Chords

If

(1) `aEqual=True`
(2) `exclude$equalChords=True`
(3) $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$
(4) $P_1P_3 = Q_1Q_3$
(5) $\odot P_1P_2P_3$ and $\odot Q_1Q_2Q_3$ are known circles
or
$\odot P_1P_2P_3$ and $\odot Q_1Q_2Q_3$ have the same center

then exclude the result $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$ as trivial (Figure 17).

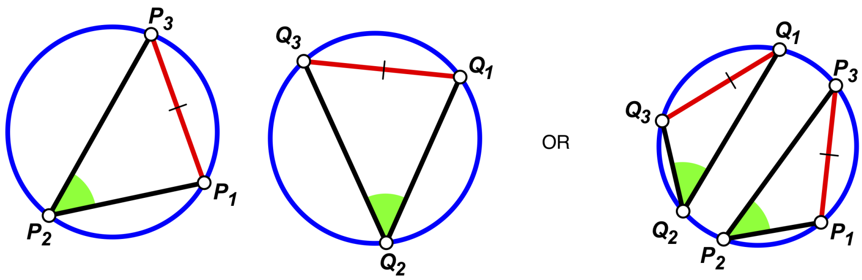


Figure 17. Detecting Equal Angles Associated with Equal Chords

Note that if two equal chords in two circles subtend equal angles, then the two circles must be congruent.

Note also that Algorithm H does not handle all instances of equal angles formed by equal chords in a circle. For example, Figure 18 shows a case that is not handled.

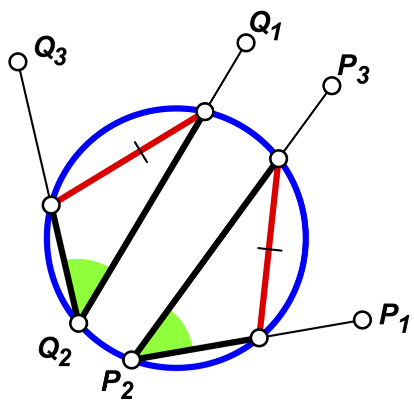


Figure 18. not yet handled

2.8. **Corresponding Angles of Similar Triangles.** It is well known that corresponding angles of similar triangles are congruent.

To exclude these equal angles as trivial, GeometricExplorer first makes a list of all similar (or congruent) triangles in the generated figure. Then it makes a table of all the pairs of equal corresponding angles (using the canonical name for the angle). To rule out a pair of equal angles as trivially equal because they are corresponding angles in similar triangles, it is only necessary to see if this pair of angles appears in the table.

2.9. **Central Angles with Equal Chords.** It is well known that equal chords in congruent circles (or in the same circle) subtend equal central angles at the center of the circle.

It is not necessary for GeometricExplorer to use this theorem to exclude trivial cases of equal central angles. For if $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$ where P_2 and Q_2 are the centers of congruent circles with equal chords P_1P_3 and Q_1Q_3 (Figure 19), then the equality of these angles will be excluded anyhow because they are corresponding angles of similar triangles. Note that $\triangle P_1P_2P_3 \cong \triangle Q_1Q_2Q_3$.

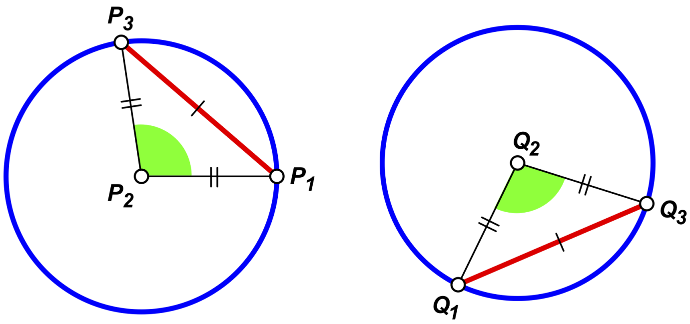


FIGURE 19. Equal Central Angles in Congrruent Circles

2.10. **Base Angles of an Isosceles Trapezoid.** It is well known that the base angles of an isosceles trapezoid are equal. For example, in Figure 20, if $P_2Q_2 \parallel P_3Q_1$ and $P_2P_3 = Q_1Q_2$, then $P_2P_3Q_1Q_2$ is an isosceles trapezoid and hence $\angle P_1P_2P_3 = \angle Q_1Q_2Q_3$.

However, it is not necessary to detect trivial equal angles in this configuration because they will be excluded anyhow because they are corresponding parts of similar triangles. Note that $\triangle Q_2P_2Q_1 \cong \triangle P_2Q_2P_3$.

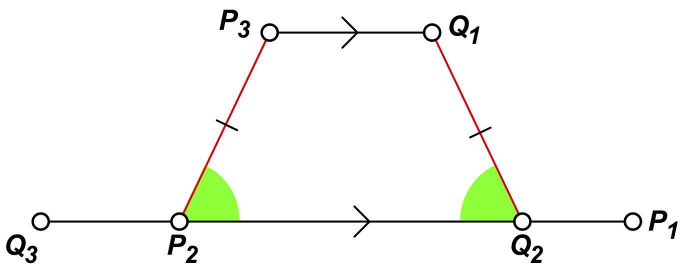


FIGURE 20. Equal Base Angles of an Isosceles Trapezoid

2.11. **Special Angles.** An angles whose measure is a multiple of $1°$ or a multiple of $\pi/7$ radians is called a *special angle*. It is interesting to discover that an angle in a geometric figure is a special angle, so GeometricExplorer will report all special angles that it finds in a figure.

Since two angles are congruent if they have the same measure, there is no point in reporting both that two angles are equal and that they are both special with the same measure. Thus, we deem it trivial that two angles are equal if they are both special, since their equality follows trivially from the fact that they have the same measure. GeometricExplorer uses the exclusion variable `exclude$specialAngles` to control whether of not equal special angles are included in reports.

Here is an example. In Figure 21, equilateral triangle BEC has been constructed inside square $ABCD$. Point F is constructed as the midpoint of DE. The conclusion is that $\angle FCE = \angle EAD$.

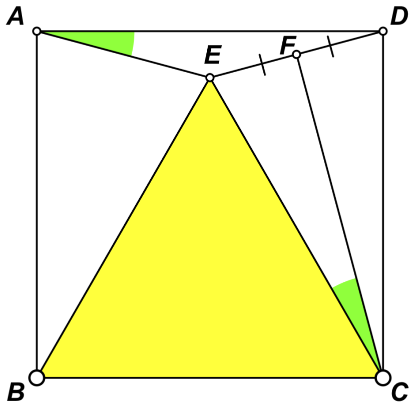


Figure 21. An Equilateral Triangle in a Square

This conclusion is not obvious. However, GeometricExplorer examines the measures of all angles in the figures and finds that both of these angles have measure 15°. The fact that $\angle FCE = 15°$ is easy to see because it is half of $\angle DCE$ which is easily determined to be 30°. On the other hand, it is not immediately obvious that $\angle EAD = 15°$.

Nevertheless, since these two angles both have measure 15°, the result $\angle FCE = \angle EAD$ will be excluded from discovery reports, since the report will contain the facts that $\angle FCE = 15°$ and $\angle EAD = 15°$.

2.12. **Right Angles.** It is well known that all right angles are congruent (Euclid Postulate 4). It is not necessary to use this result when deciding to exclude two equal right angles, because the equality of the angles would be excluded by the fact that the two angles are both special angles with the same measure.

2.13. **Construction Exclusions.** There are some angles that are trivially equal because of the way that they were constructed. For example, if one of the steps used in creating a figure is to draw the angle bisector of a given angle, then trivially, the two angles formed by the angle bisector and the sides of the angle are equal. It is not possible to determine that these angles are trivially equal merely by examining the final figure, without knowing how the figure was constructed.

Thus, GeometricExplorer detects such trivial results using a different mechanism. When constructions are applied to a figure, if a construction causes some trivial result to hold, then this result is entered into a table of *known results*. Later, when scanning a figure for interesting properties, if a property is found, it is checked to see if that property is listed amongst the known results. If the property is so listed, then the property is excluded from reports because it is a trivial property.

For example, if one step of a sequence of constructions is to draw the median of a triangle, then we not that the endpoint of the median divides the side to which it is drawn into two equal segments. The equality of these two segments is then entered into the table of known results so that it can later be excluded as a trivial result.

Such exclusions are known as *construction exclusions* to differentiate them from *property exclusions* that have been discussed previously.

We now list some standard constructions that trivially cause two angles in the generated figure to be equal.

2.13.1. *Angle Bisectors.*

Clearly, when the angle bisector of an angle is constructed, the two constructed angles are equal and this result should be excluded as trivial.

Construction Exclusion A.
Note Equal Angles Formed when Constructing an Angle Bisector
If

(1) `aEqual=True`
(2) `exclude$angleBisectorProperties=True`
(3) Angle bisector AD of $\angle BAC$ is constructed

then note the result $\angle BAD = \angle DAC$ (using canonical names for the angles) as a known result (Figure 22).

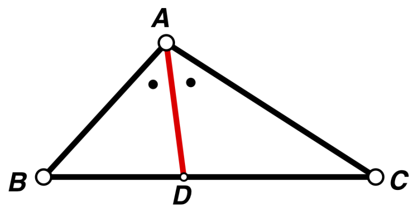


FIGURE 22. Equal angles formed by an angle bisector

2.13.2. *Incenter/Excenter.*

Constructing the incenter or an excenter of a triangle results in various angles being equal. These properties are well known and should be excluded as trivial.

Construction Exclusion I.
Note Equal Angles Formed when Constructing an Incenter
If

(1) `aEqual=True`
(2) `exclude$incenterProperties=True`
(3) The incenter I of $\triangle ABC$ is constructed

then note the following results (using canonical names for the angles) as known results (Figure 23).

(a) $\angle BAI = \angle IAC$
(b) $\angle CBI = \angle IBA$
(c) $\angle ACI = \angle ICB$

The same construction exclusion is used when constructing the triangle center X_1 of a triangle or when constructing a center whose first trilinear coordinate is a nonzero constant.

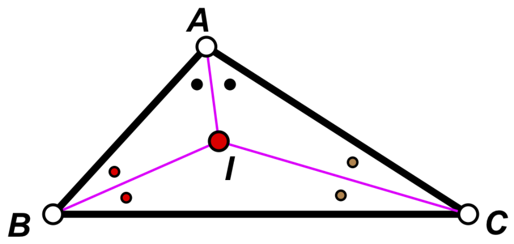


Figure 23. Equal angles formed by an incenter

Construction Exclusion E.
Note Equal Angles Formed when Constructing an Excenter
If

(1) `aEqual=True`
(2) `exclude$excenterProperties=True`
(3) The excenter J of $\triangle ABC$ opposite vertex B is constructed

then note the result $\angle CBJ = \angle JBA$ (using canonical names for the angles) as a known result (Figure 24).

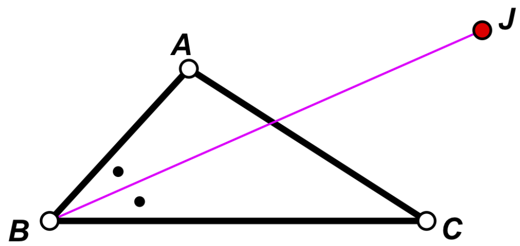


Figure 24. Equal angles formed by an excenter

2.13.3. *Apollonius Constructions.*

Some of the Apollonius constructions create equal angles that should be excluded.

Construction Exclusion LLL.
Note Equal Angles Formed when performing an LLL Construction
If

(1) `aEqual=True`
(2) `exclude$LLLProperties=True`
(3) The center I of a circle touching three known lines is constructed
(4) Two of the known lines meet at a known point P
(5) Let F_1 and F_2 be the feet of the perpendiculars from I to the two lines
(6) Angles IPF_1 and IPF_2 have canonical names

then note the result $\angle IPF_1 = \angle IPF_2$ (using canonical names for the angles) as a known result (Figure 25 left).

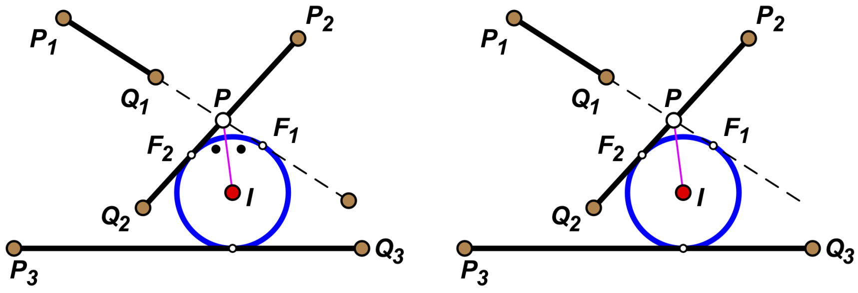


FIGURE 25. Circle tangent to three lines

Note that an angle may not have a canonical name if there is no known point on one of the rays bounding the angle. For example, in (Figure 25 right), there is no canonical name for $\angle IPF_1$ because there is no known point on ray $\overrightarrow{PF_1}$.

It should also be noted that if a known point later gets constructed on ray $\overrightarrow{PF_1}$, we will have no way of reconsidering whether or not $\angle IPF_1 = \angle IPF_2$ should be listed in a discovery report.

Here is a specific example. In Figure 26 we have started with a quadrilateral $ABCD$. We then do an LLL construction to construct a circle tangent to AB, BC, and CD. This constructs point E, the center of the circle. The LLL code notices that the two lines AB and CD do not meet at a known point, so cannot note any equal-angle property as a known property.

In a subsequent step, we do an intersection construction and construct point F, the intersection of AB and CD. This now forms two equal angles, $\angle CFE$ and $\angle EFA$. Their equality is fairly trivial, since EF bisects $\angle CFA$, but we don't catch it as trivial. So $\angle CFE = \angle EFA$ will be reported as a discovery.

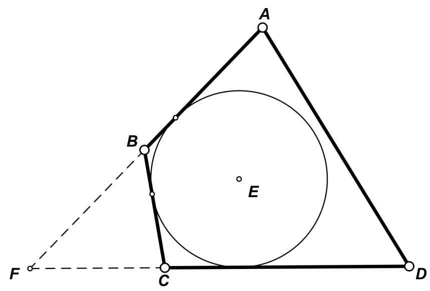


FIGURE 26. Circle tangent to three sides of a quadrilateral

Construction Exclusion LLP.
Note Equal Angles Formed when performing an LLP Construction

If

(1) `aEqual=True`
(2) `exclude$LLPProperties=True`
(3) The center I of a circle touching two known lines P_1Q_1 and P_2Q_2 and passing through a known point A is constructed
(4) The two known lines meet at a known point P
(5) Let F_1 and F_2 be the feet of the perpendiculars from I to the two lines

(6) Angles IPF_1 and IPF_2 have canonical names

then note the result $\angle IPF_1 = \angle IPF_2$ (using canonical names for the angles) as a known result (Figure 27).

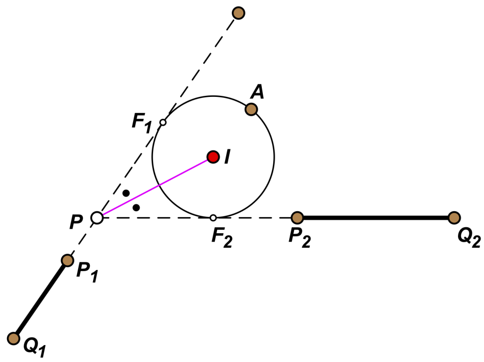


FIGURE 27. Circle tangent to two lines and passing through a point

In the example shown in (Figure 27), the result $\angle IPF_1 = \angle IPF_2$ is noted (using their canonical names). If there were no known (brown) point on ray $\overrightarrow{PF_1}$, then no result would be noted.

Construction Exclusion LLC.
Note Equal Angles Formed when performing an LLC Construction

If

(1) `aEqual=True`
(2) `exclude$LLCProperties=True`
(3) The center I of a circle touching two known lines P_1Q_1 and P_2Q_2 and a known circle C is constructed
(4) The two known lines meet at a known point P
(5) Let F_1 and F_2 be the feet of the perpendiculars from I to the two lines
(6) Angles IPF_1 and IPF_2 have canonical names

then note the result $\angle IPF_1 = \angle IPF_2$ (using canonical names for the angles) as a known result (Figure 28).

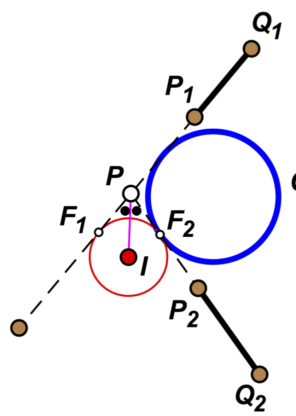


FIGURE 28. Circle tangent to two lines and a circle

2.13.4. *Orthocenter.*

Construction Exclusion H.
Note Equal Angles Formed when Constructing an Orthocenter

If

(1) `aEqual=True`
(2) `exclude$OrthocenterProperties=True`
(3) The orthocenter H of $\triangle ABC$ is constructed
(4) The point H lies inside $\triangle ABC$

then note the following results (using canonical names for the angles) as known results (Figure 29).

(a) $\angle ACH = \angle HBA$
(b) $\angle BAH = \angle HCB$
(c) $\angle CBH = \angle HAC$

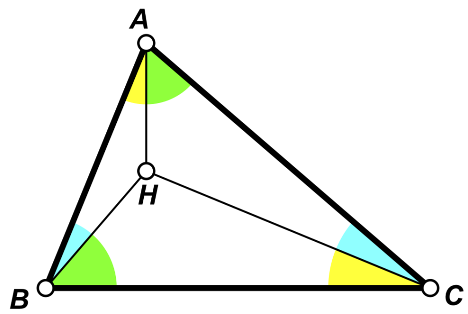


FIGURE 29. Equal angles formed by an orthocenter

The same construction exclusion is used when constructing the triangle center X_4 of a triangle.

3. CORE PROPERTIES

Recall that GeometricExplorer will exclude the result $\angle CBA = \angle ACB$ in Figure 30 because $AB = AC$ and the base angles of an isosceles triangle are equal.

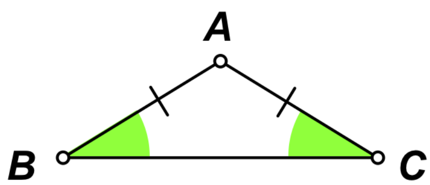
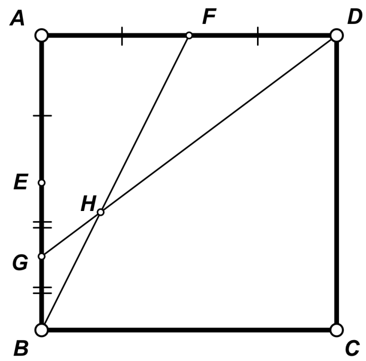


FIGURE 30. Base angles of an isosceles triangle

But what if it is not obvious that $AB = AC$? For example, from the way Figure 1 (shown again to the right) was constructed, it is not obvious that $GH = GB$. The fact that these two line sements are equal is a fact that GeometricExplorer discovers when examining the figure. If a user asks GeometricExplorer to make a report of nontrivial equal angles in this figure, should the fact that $\angle FBA = \angle GHB$ be included in the report?



When GeometricExplorer makes a report about properties found in a figure, the report is often very long. In order for a user to spot new discoveries, it is important to weed out as many trivial results as possible. We therefore exclude the property $\angle FBA = \angle GHB$ because these angles are base angles of an isosceles triangle. In this case, the result is not obvious because the fact that $GH = GB$ is not obvious. However, this nontrivial result will not be lost because the fact that $GH = GB$ will be included in a report listing all equal line segments in the figure. It would be redundant to list both $\angle FBA = \angle GHB$ and $GH = GB$ in a report of results discovered in the figure.

We therefore make the following rule. Whenever an angle relationship follows trivially from some other core property, then the angle relationship will be excluded from reports. The *core properties* that GeometricExplorer examines are the following.

(1) points A and B coincide
(2) points A, B, and C are collinear
(3) lines L_1, L_2, and L_3 concur
(4) $AB = CD$
(5) $AB \parallel CD$
(6) $AB \perp CD$
(7) points A, B, C, and D are concylic
(8) $\triangle ABC \sim \triangle XYZ$
(9) the measure of $\angle ABC$ is an integral multiple of 1° or $\pi/7$

Here is another example. In Figure 31, we start with a right triangle ABC with right angle at C. Then we draw the altitude to the hypotenuse CE. Constructing the midpoint D of AE and the midpoint F of CE leads to the figure. The conclusion is that $\angle BDC = \angle EFB$.

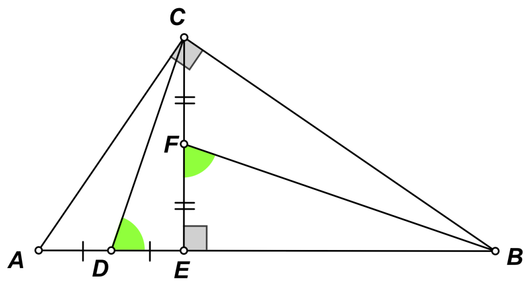


Figure 31. Green angles are equal

This is not a trivial or obvious conclusion, so at first glance, one would expect GeometricExplorer to report this as an interesting discovery. However, GeometricExplorer looked at all pairs of triangles in Figure 31 and found that triangles EDC and EFB are similar. This too is not immediately obvious. The fact that $\triangle EDC \sim \triangle EFB$ means that $\angle BDC = \angle EFB$ because they are corresponding angles in the similarity. So which result should be reported?

Since similarity of triangles is a core property, this means that we will report the similarity and then exclude the angle equality because it follows trivially from the similarity.

4. Transitivity

Another useful technique for excluding extraneous relationships from discovery reports is the concept of transitivity. If it has been determined that $\angle A = \angle B$ in a figure is well known, trivial, or obvious and also that $\angle B = \angle C$ is well known, trivial, or obvious, then we can conclude that $\angle A = \angle C$ is true and follows trivially from the first two equalities. Thus, $\angle A = \angle C$ should also be excluded from discovery reports.

Here is an example. In figure 32, we started with $\triangle ABC$ and constructed the incircle. It touches side BC at D. Then we constructed the mixtilinear incircle opposite A (red) which is tangent to sides AB and AC of the triangle and tangent to its circumcircle at point E.

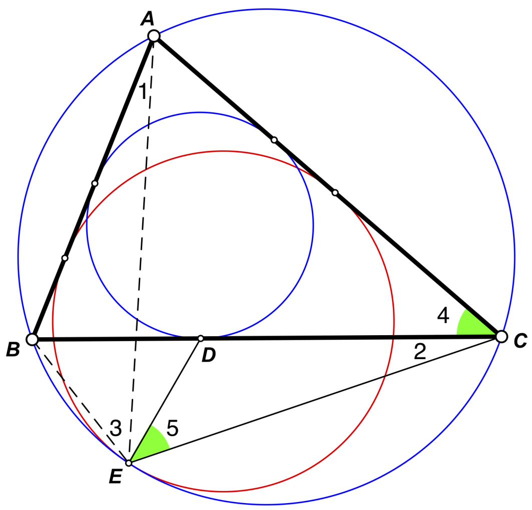


Figure 32. Green angles are equal due to transitivity

Upon analyzing the figure, we find the interesting discovery that $\triangle CED \sim \triangle AEB$ which leads us to the fact that $\angle 5 = \angle 3$. We also note that $\angle 3 = \angle 4$ by Algorithm Q (they subtend the same arc of the circumcircle). By transitivity of angle equality, we can therefore conclude that $\angle 5 = \angle 4$.

Although this is not obvious, it follows immediately by transitivity from the first two results, so the relationship $\angle 5 = \angle 4$ will be excluded from discovery reports. The control variable `aUseTransitivity` is used to enable or disable this feature.

The transitivity property can be extended to longer chains of equality. If $\angle 1 = \angle 2$ trivially and $\angle 2 = \angle 3$ trivially, and so on until $\angle(n-1) = \angle n$ trivially, then by transitivity, we can conclude that $\angle 1 = \angle n$ trivially and should be excluded from discovery reports.

In order to test for exclusions based on transitivity, we create a triviality graph. Each node in the graph corresponds to an angle in the figure. An arc is drawn from node A to node B if the angle represented by node A has been determined to be trivially equal to the angle represented by node B. We then compute the transitive closure of this graph. Any pair of angles then connected by an arc is deemed trivial and excluded from discovery reports.

Let us use Figure 1 as an example. It is repeated here as Figure 33

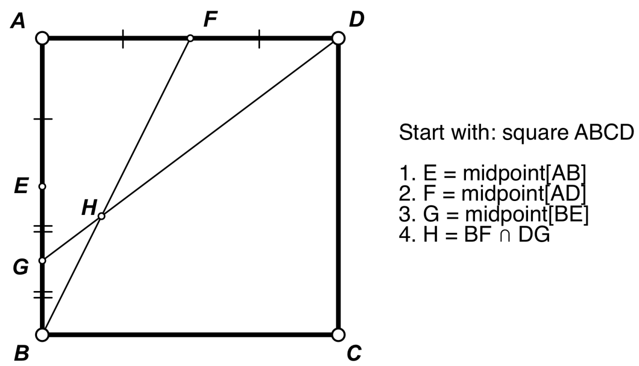


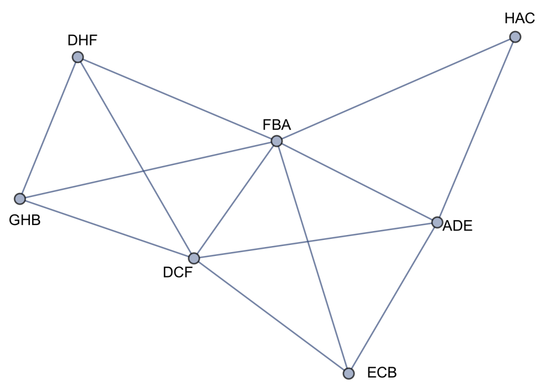
Figure 33. A square with Four Constructions

There are 87 angles in this figure. The transitivity graph consists of 12 subgraphs and is shown in Figure 34. Each subgraph consists of angles with the same measure. The vertices of the graph represent the angles. A line joining two vertices means that the equality of these two angles has been deemed trivial by one of the algorithms given in this paper.

For example, in the last subgraph, angles FHG and BHD are connected by a line segment. This means that the two angles are trivially equal. In this case, they are equal because they are vertical angles. In subgraph 11, angles ADG and FCE are trivially equal because they are corresponding angles of similar triangles ADG and HCF. Note that C, H, and E are collinear. Also note that C, D, F, and H are concyclic, so CF is a diameter of $\odot CDFH$ and thus $\angle CHF$ is a right angle.

Now let us look more closely at subgraph 10, enlarged below.

This subgraph consists of 7 angles that are equal. Again, a line joining two angles means that they are trivially equal, by one of our exclusion criteria. For example, there is a line joining $\angle DHF$ and $\angle GHB$ in the graph because they are vertical angles. The line joining $\angle GHB$ and $\angle FBA$ is because these two angles are base angles of isosceles triangle BGH (GeometricExplorer determined that $GH = GB$).



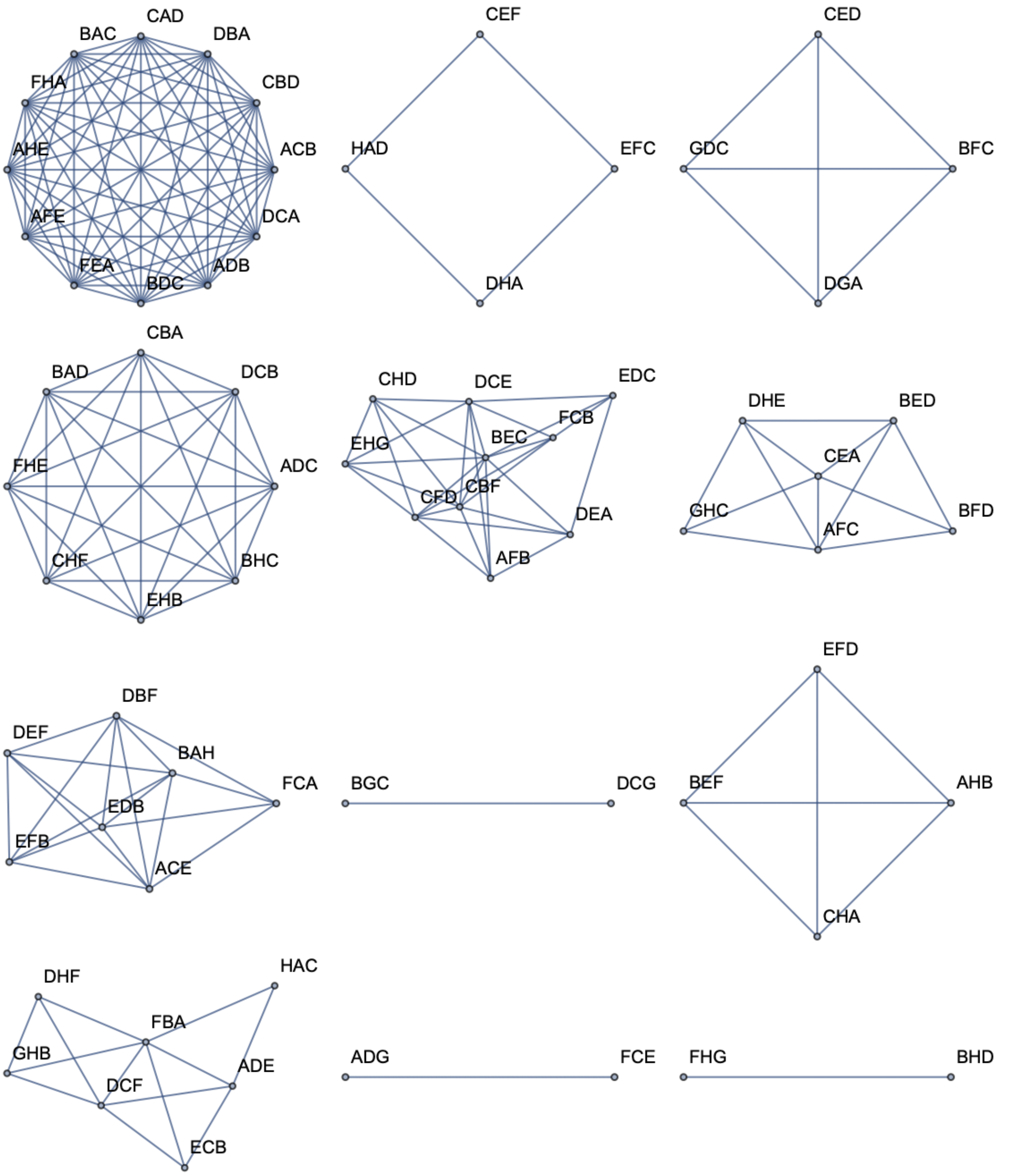


FIGURE 34. Graph corresponding to Figure 33

There is no line joining $\angle GHB$ and $\angle ADE$ because these two angles are not trivially equal by any of our exclusion criteria. However, since there is a path from $\angle GHB$ to $\angle ADE$ in the graph, this means that these two angles are trivially equal by our transitivity rule. In fact, subgraph 10 is a connected graph. Thus all the angles represented by this subgraph are considered trivially equal and will be excluded from discovery reports.

If the subgraph had not been connected, for example, if it had two disjoint connected components, then the equality between one angle in one component and one angle in the other component would be a nontrivial discovery. Looking at each of the 12 subgraphs, we notice that they are all connected. Thus, when GeometricExplorer analyzes Figure 33, it does not report any equal-angle discoveries.

5. Double Angles

There are also some cases where the result that one angle has twice the measure of another angle is trivial and should be excluded.

Algorithm DA. Detect Double Angles Associated with an Angle Bisector

If

(1) `aRatioRational=True`
(2) `exclude$angleBisectorProperties=True`
(3) $\angle P_1P_2P_3 = 2\angle Q_1Q_2Q_3$
(4) $P_1 = Q_1$ and $P_2 = Q_2$

then exclude the result $\angle P_1P_2P_3 = 2\angle Q_1Q_2Q_3$ as trivial (Figure 35).

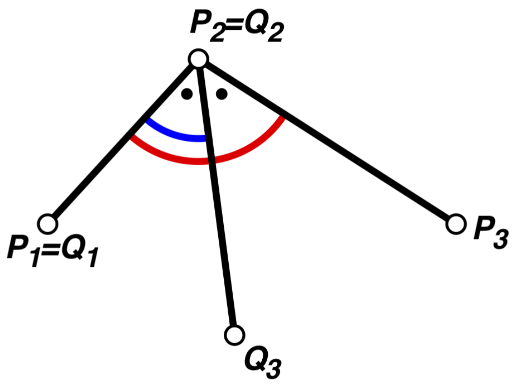


Figure 35. Red angle is twice blue angle.

In the same manner, we can exclude the result $\angle P_1P_2P_3 = 2\angle Q_3Q_2P_3$.

Note that the control variable `aRatioRational` is used to determine whether or not to check for the measure of one angle being a rational multiple of the measure of another angle.

Algorithm D1. Detect Double Angles Associated with an Isosceles Triangle

If

(1) `aRatioRational=True`
(2) `exclude$isoscelesTriangles=True`
(3) $\angle P_1P_2P_3 = 2\angle Q_1Q_2Q_3$
(4) P_2, P_3, Q_2, Q_3 are collinear
(5) $P_1P_2 \cap Q_1Q_2 = X$ and X is a known point in the figure
(6) $P_2Q_2 = P_2X$

then exclude the result $\angle P_1P_2P_3 = 2\angle Q_1Q_2Q_3$ as trivial (Figure 36).

Algorithm D2. Detect Double Angles Associated with an Isosceles Triangle

If

(1) `aRatioRational=True`
(2) `exclude$isoscelesTriangles=True`

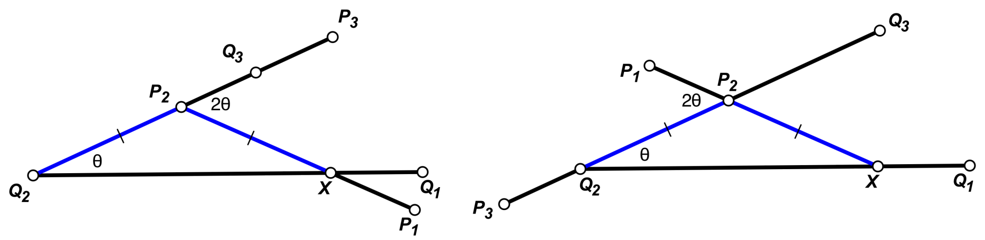


FIGURE 36. Detecting Double Angles Associated with an Isosceles Triangle

(3) $\angle P_1P_2P_3 = 2\angle Q_1Q_2Q_3$
(4) P_1, P_2, Q_1, Q_2 are collinear
(5) $P_2P_3 \cap Q_2Q_3 = X$ and X is a known point in the figure
(6) $P_2Q_2 = P_2X$

then exclude the result $\angle P_1P_2P_3 = 2\angle Q_1Q_2Q_3$ as trivial (Figure 37).

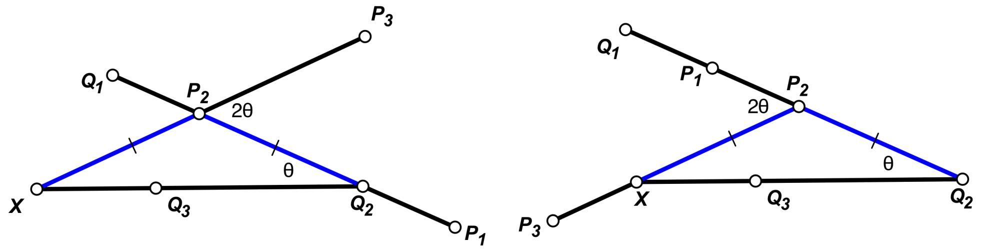


FIGURE 37. Detecting Double Angles Associated with an Isosceles Triangle

Algorithm C. Detect Double Angles Associated with a Central Angle

If

(1) `aRatioRational=True`
(2) `exclude$centralAngles=True`
(3) $\angle P_1P_2P_3 = 2\angle Q_1Q_2Q_3$
(4) $P_1P_2 \cap Q_1Q_2 = X$ and X is a known point in the figure
(5) $P_2P_3 \cap Q_2Q_3 = Y$ and Y is a known point in the figure
(6) $P_2X = P_2Y = P_2Q_2$

then exclude the result $\angle P_1P_2P_3 = 2\angle Q_1Q_2Q_3$ as trivial (Figure 38).

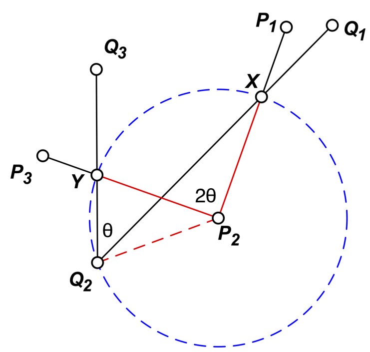


FIGURE 38. Detecting Double Angles Associated with Central Angles

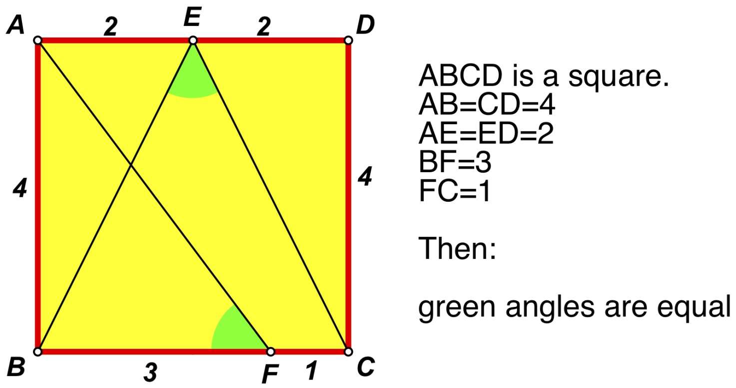
6. New Results

After seeing all the equal angle exclusions described above, the reader may wonder if there are any nontrivial results showing that two angles are equal. There are! We present here a selection of results involving equal angles found using GeometricExplorer. If an elegant geometric proof is known, we give it; otherwise, we give a reference to the literature where the proof can be found.

6.1. **Squares.**

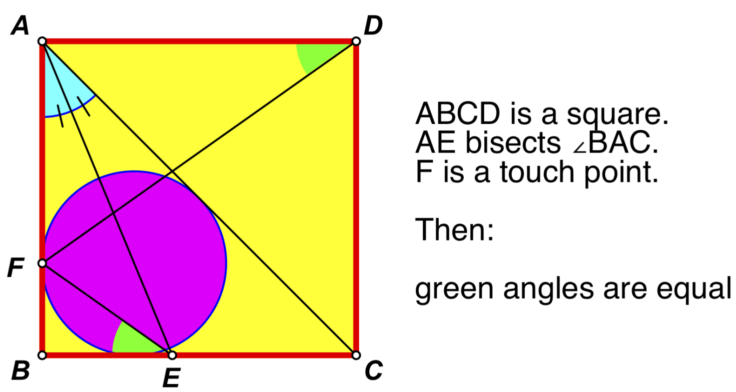
We found the following results involving pairs of equal angles associated with squares.

Theorem 1.



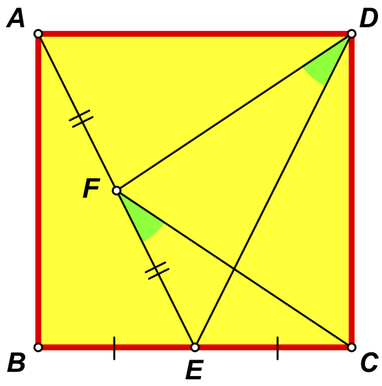
Proof. See [24]. □

Theorem 2.



Proof. See [22]. □

Theorem 3.



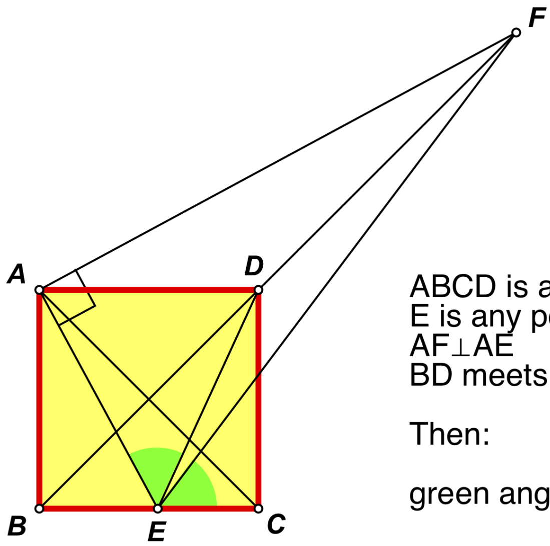
ABCD is a square.
BE=EC.
AF=FE.

Then:

green angles are equal

Proof. See [23]. □

Theorem 4.



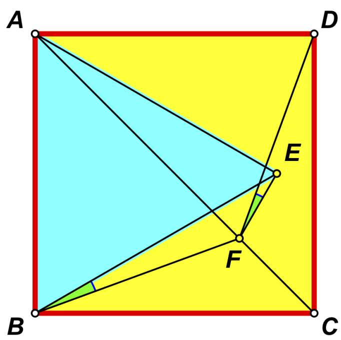
ABCD is a square.
E is any point on BC.
AF⊥AE
BD meets AF at F.

Then:

green angles are equal

Proof. See [21]. □

Theorem 5.



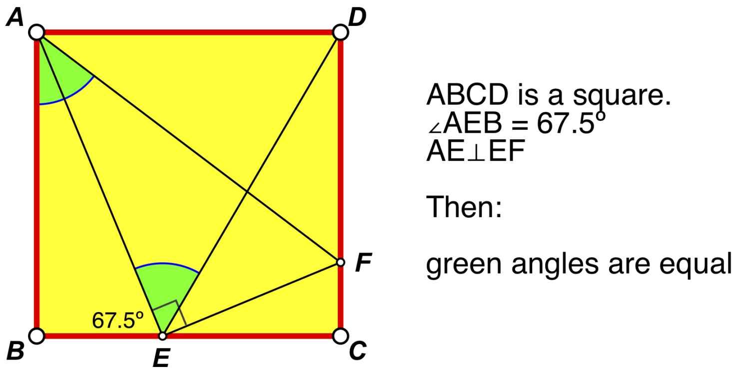
ABCD is a square.
ABE is an equilateral triangle
AE⊥EF

Then:

green angles are equal

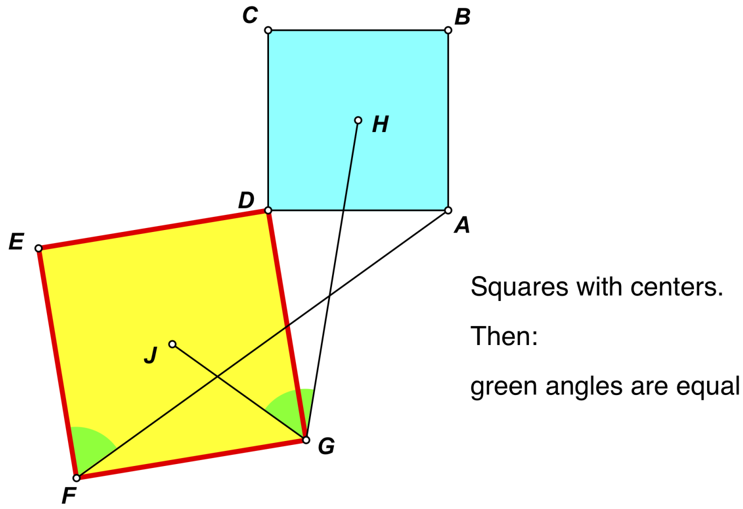
Proof. See [15]. □

Theorem 6.



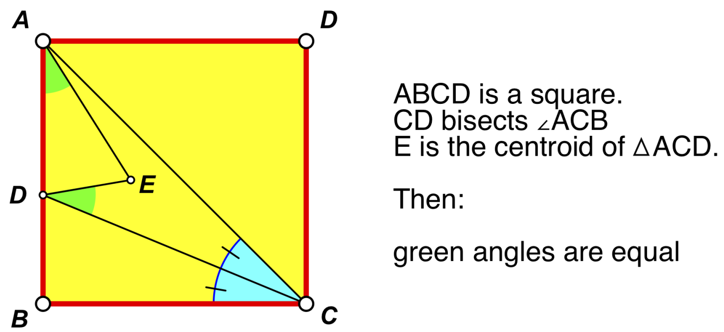
Proof. See [16]. □

Theorem 7.



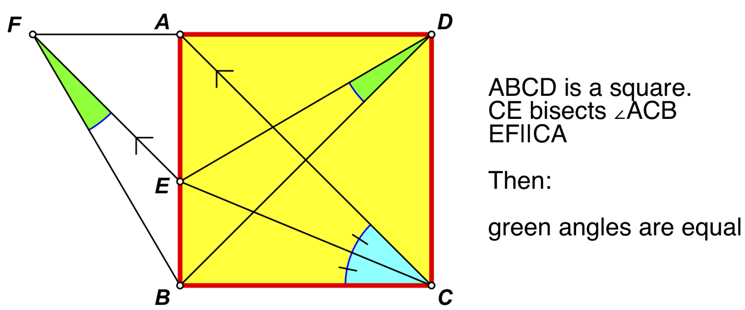
Proof. See [43]. □

Theorem 8.



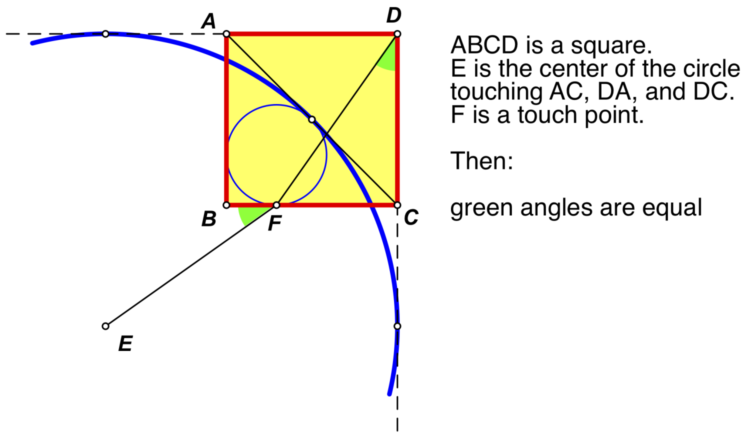
Proof. See [20]. □

Theorem 9.



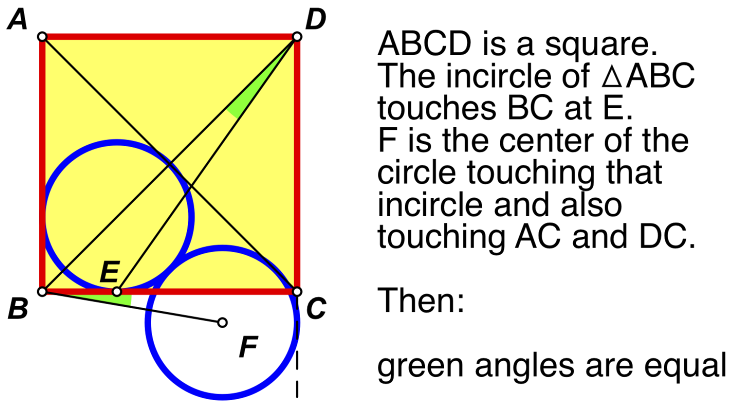
Proof. See [14]. □

Theorem 10.



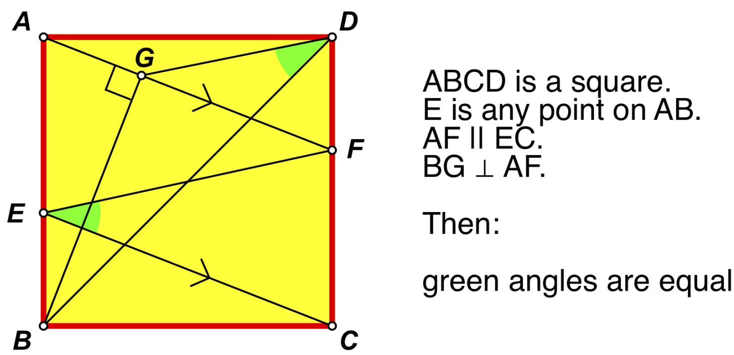
Proof. See [83]. □

Theorem 11.



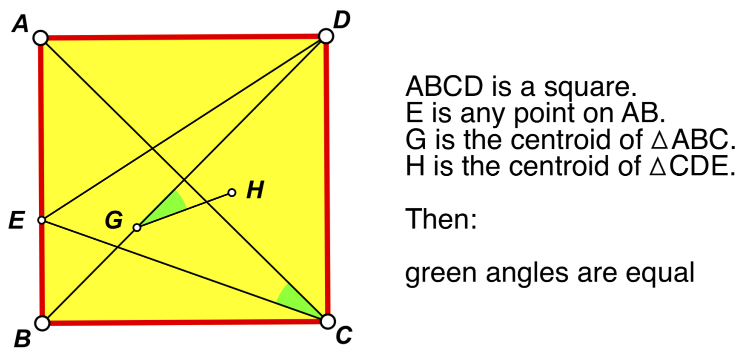
Proof. See [84]. □

Theorem 12.



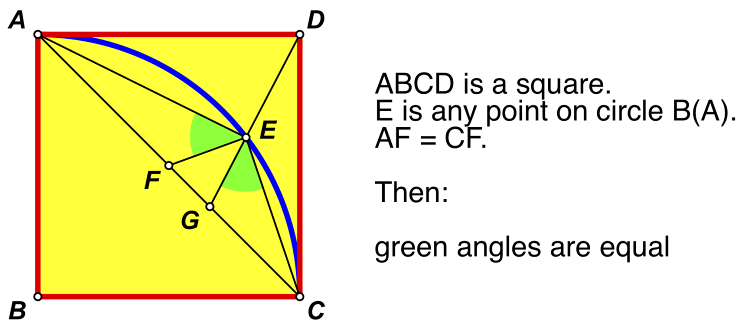
Proof. See [52]. □

Theorem 13.



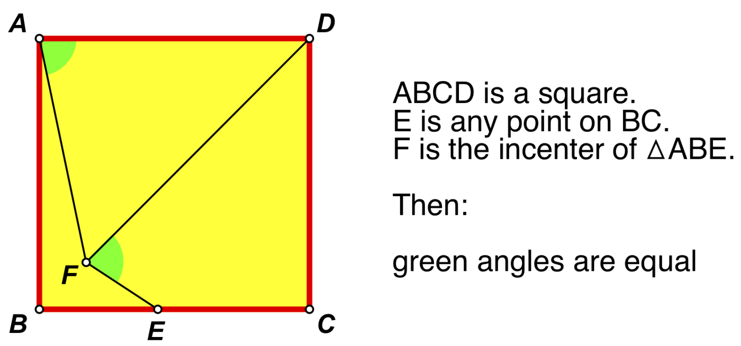
Proof. See [54]. □

Theorem 14.



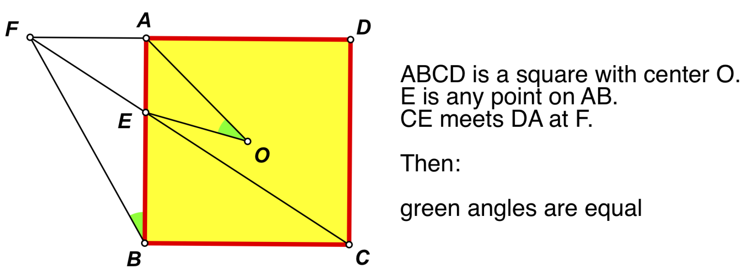
Proof. See [56]. □

Theorem 15.



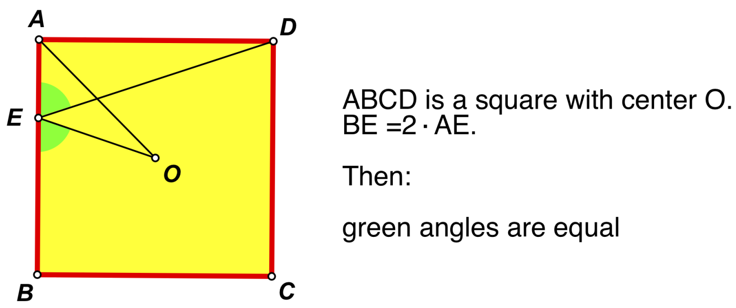
Proof. See [57]. □

Theorem 16.



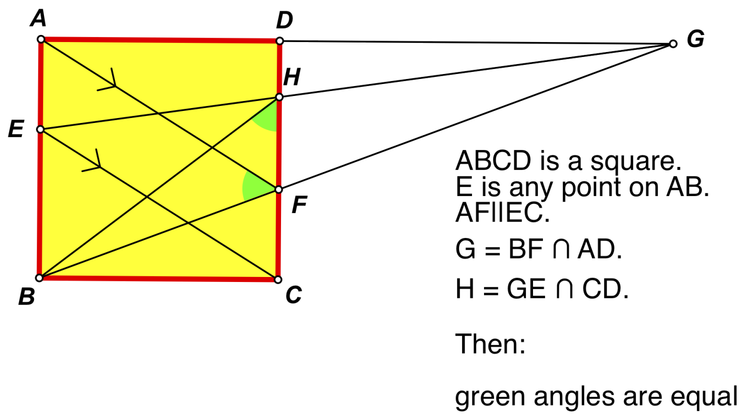
Proof. See [58]. □

Theorem 17.



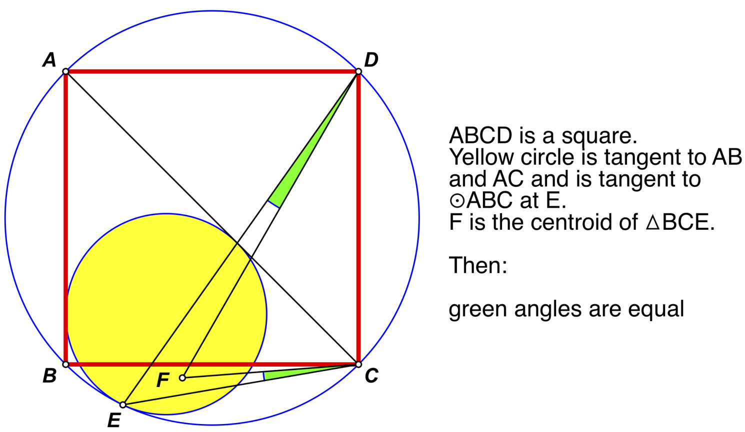
Proof. See [59]. □

Theorem 18.



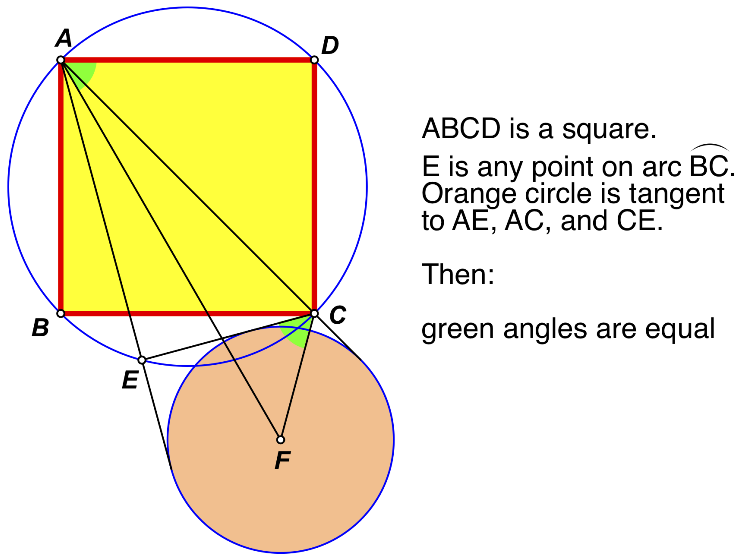
Proof. See [60]. □

Theorem 19.



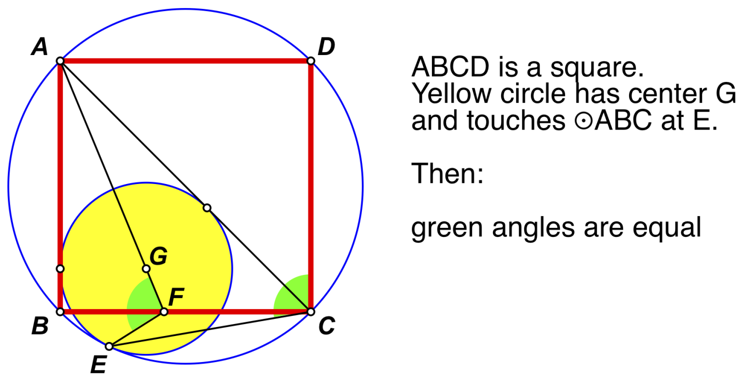
Proof. See [63]. □

Theorem 20.



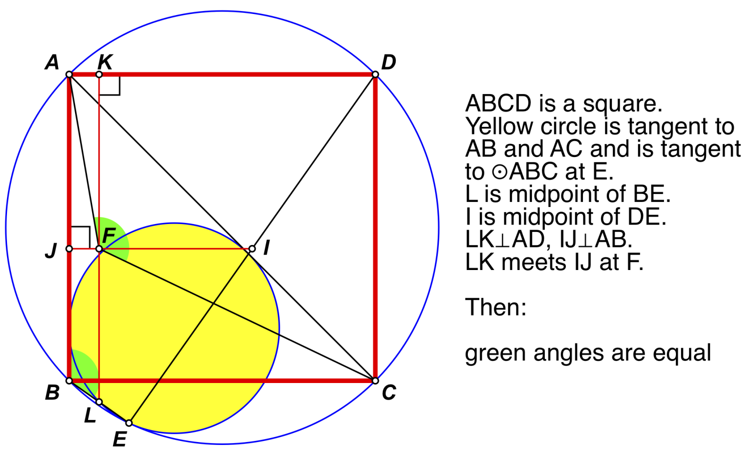
Proof. See [64]. □

Theorem 21.



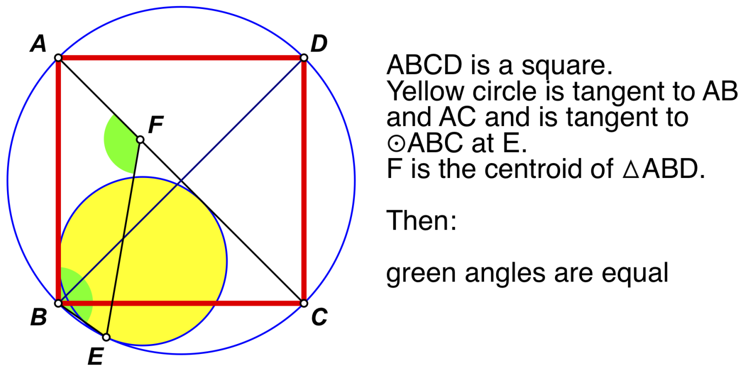
Proof. See [65]. □

Theorem 22.



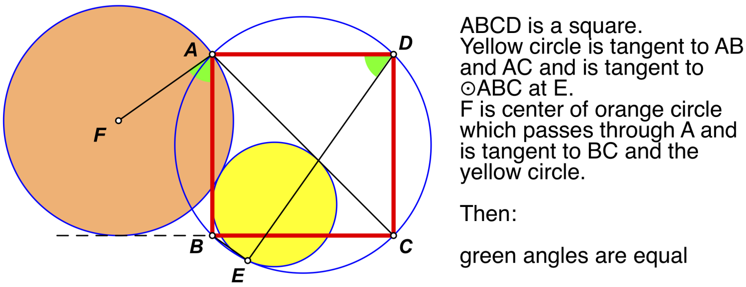
Proof. See [66]. □

Theorem 23.



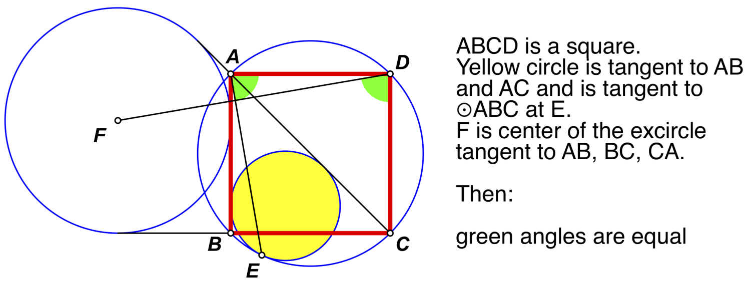
Proof. See [67]. □

Theorem 24.



Proof. See [68]. □

Theorem 25.

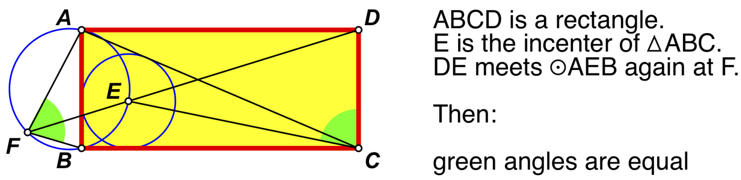


Proof. See [69]. □

6.2. **Rectangles.**

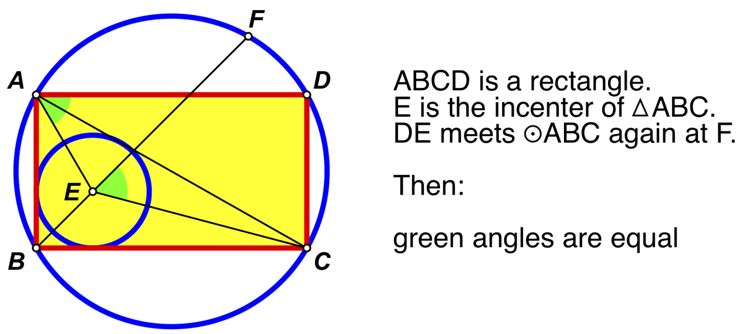
We found the following results involving pairs of equal angles associated with rectangles.

Theorem 26.



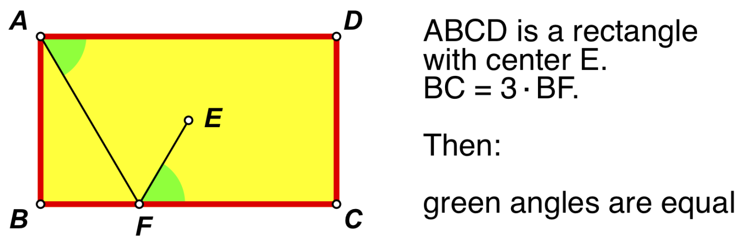
Proof. See [87]. □

Theorem 27.



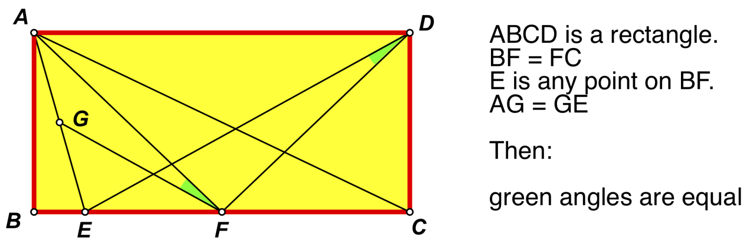
Proof. See [88]. □

Theorem 28.



Proof. See [89]. □

Theorem 29.

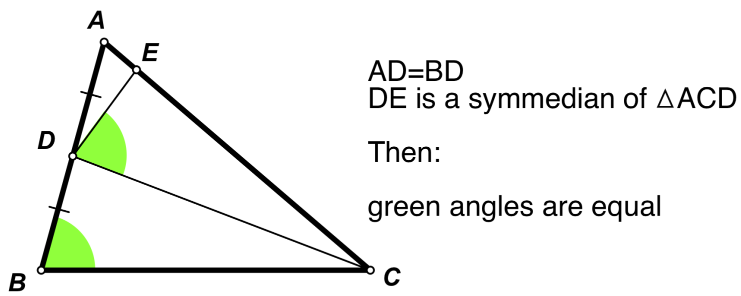


Proof. See [92]. □

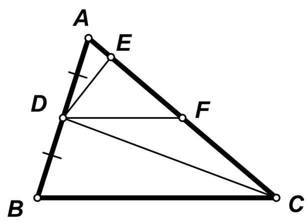
6.3. **Symmedians.**

We found the following results involving pairs of equal angles associated with symmedians.

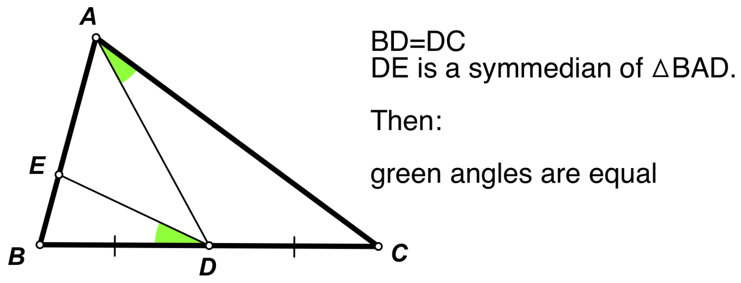
Theorem 30.



Proof. Let F be the midpoint of AC, so that DF is a median of $\triangle ACD$. From the definition of a symmedian, $\angle EDA = \angle GDF$. Since D and F are midpoints, $DF \parallel BC$ and hence $\angle CBA = \angle FDA = \angle FDE + \angle EDA = \angle FDE + \angle CDF = \angle CDE$. □

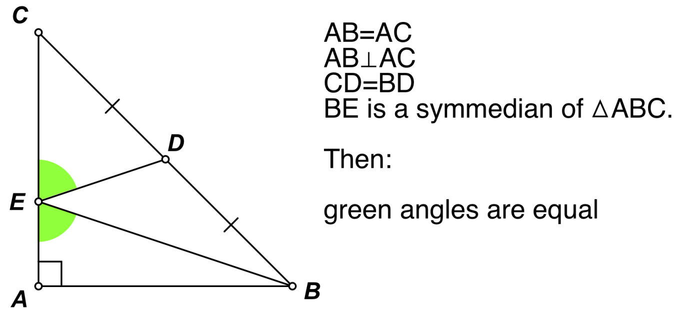


Theorem 31.



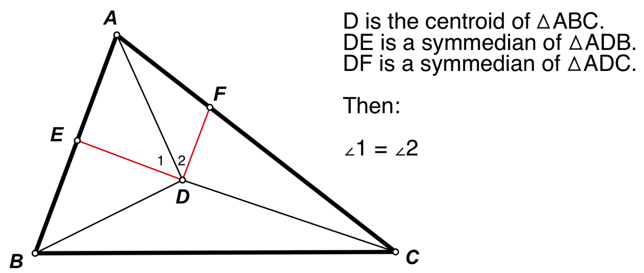
Proof. See [10]. □

Theorem 32.



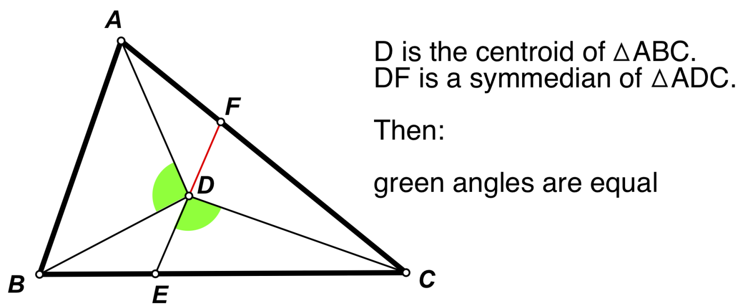
Proof. See [19]. □

Theorem 33.



Proof. See [85]. □

Theorem 34.

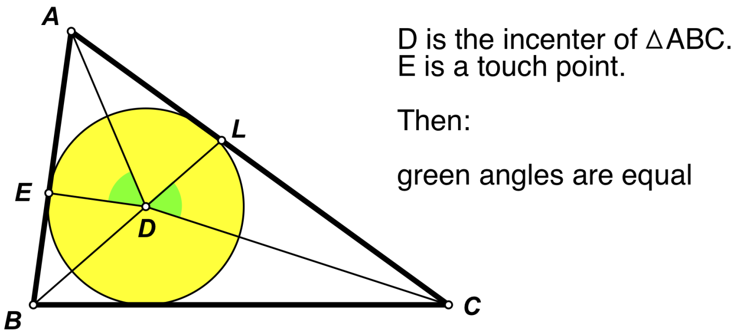


Proof. See [86]. □

6.4. Incircles.

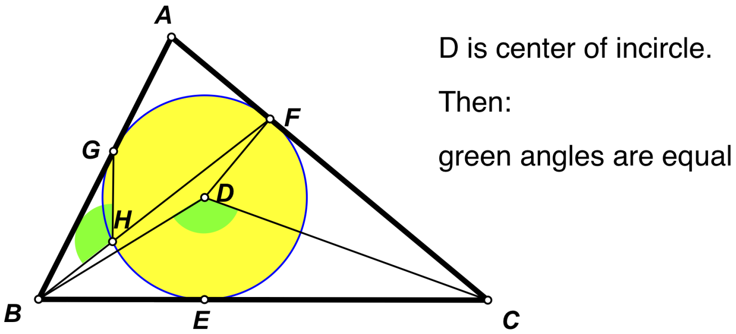
We found the following results involving pairs of equal angles associated with incircles.

Theorem 35.



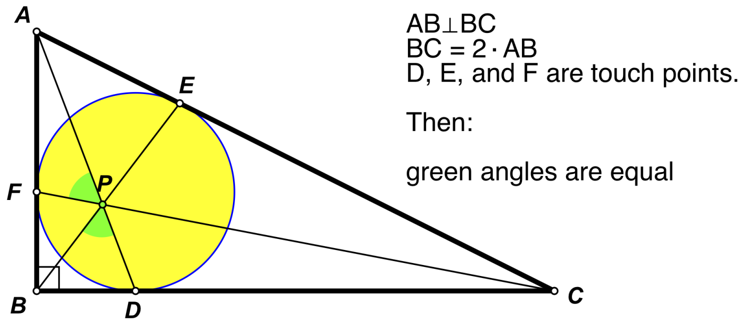
Proof. $\angle ADE = 90° - \angle EAD = (180° - \angle BAC)/2 = (\angle CBA + \angle ACB)/2 = \angle CBD + \angle DCB = \angle CDL$. See also [55]. □

Theorem 36.



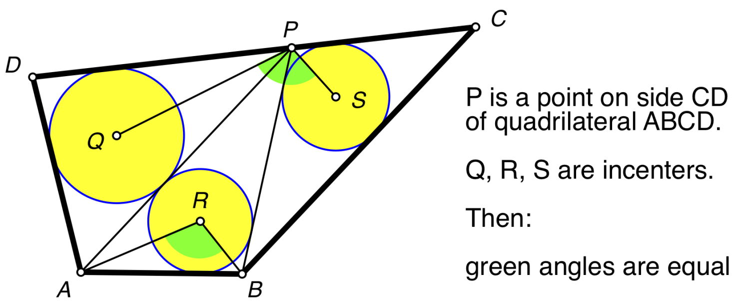
Proof. See [62]. □

Theorem 37.



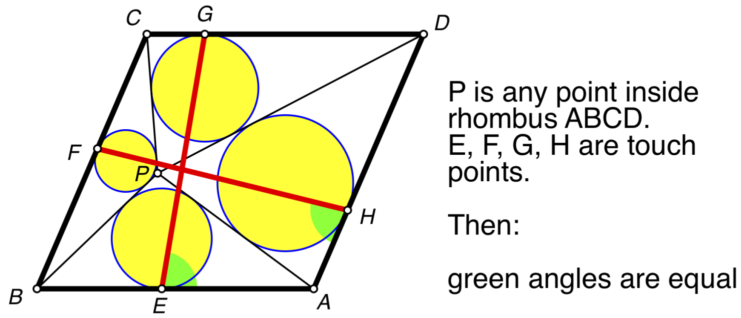
Proof. See [77]. □

Theorem 38.



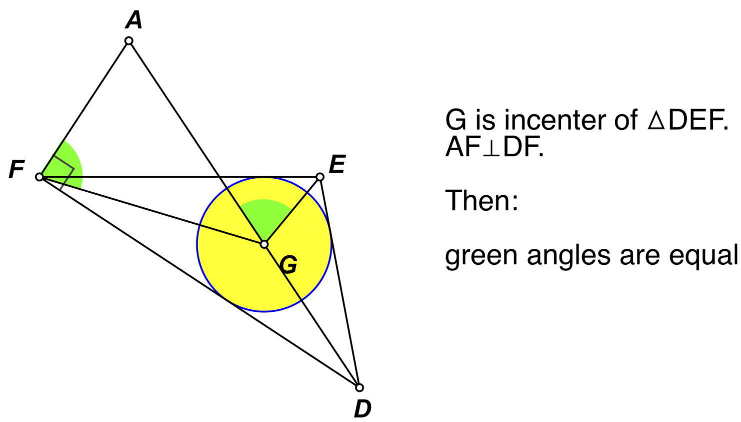
Proof. See [81]. □

Theorem 39.



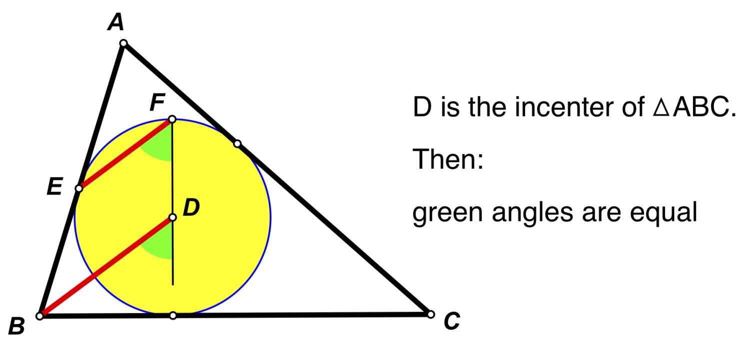
Proof. See [82]. □

Theorem 40.



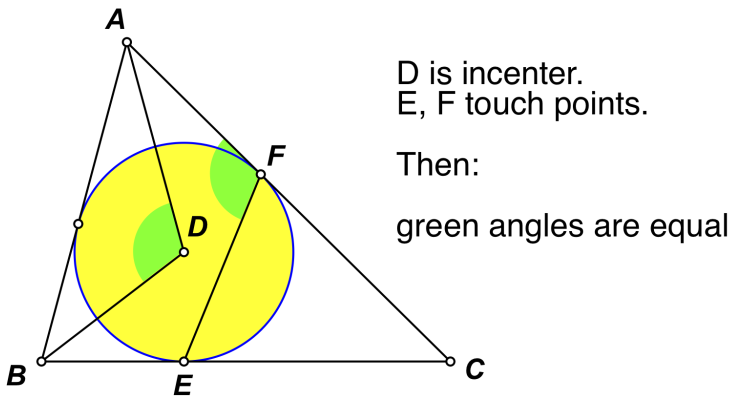
Proof. See [35]. □

Theorem 41.



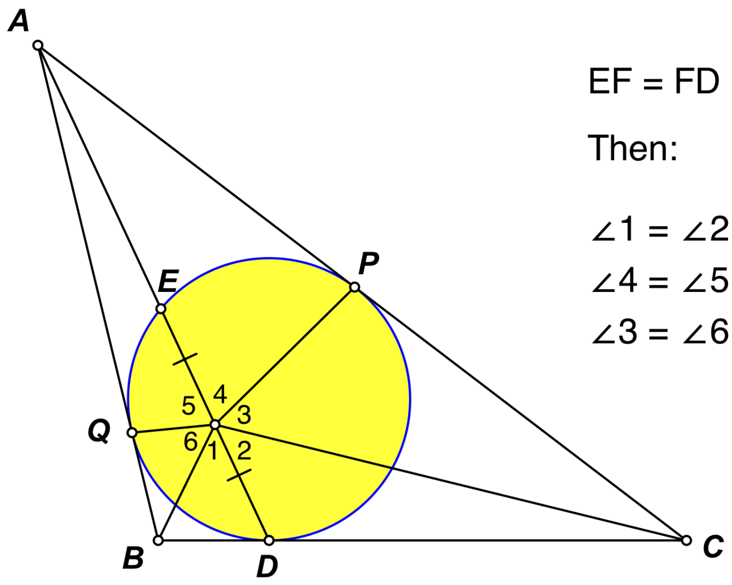
Proof. See [61]. □

Theorem 42.



Proof. See [53]. □

Theorem 43.

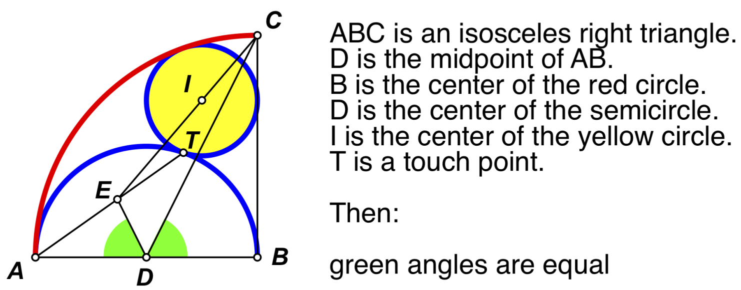


Proof. See [42]. □

6.5. Circles.

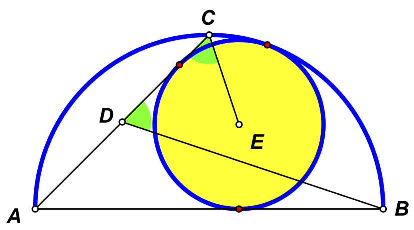
We found the following results involving pairs of equal angles associated with circles.

Theorem 44.



Proof. See [91]. □

Theorem 45.



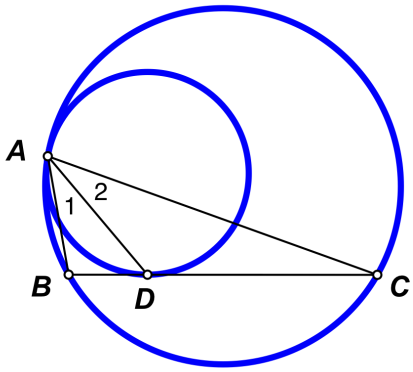
C is the midpoint of the arc of a semicircle with diameter AB.
D is the midpoint of AC.
E is the center of the circle touching AB, AC, and the semicircle.

Then:

green angles are equal

Proof. See [90]. □

Theorem 46.

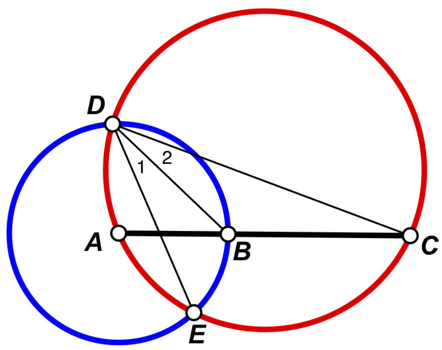


The two circles are tangent at A.
D is a touch point.

Then: ∠1 = ∠2

Proof. See [80]. □

Theorem 47.

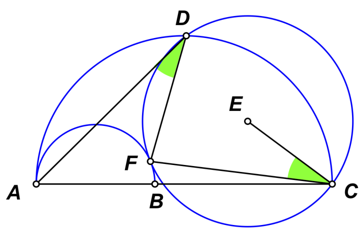


A is the center of the blue circle.

Then: ∠1 = ∠2.

Proof. See [73]. □

Theorem 48.



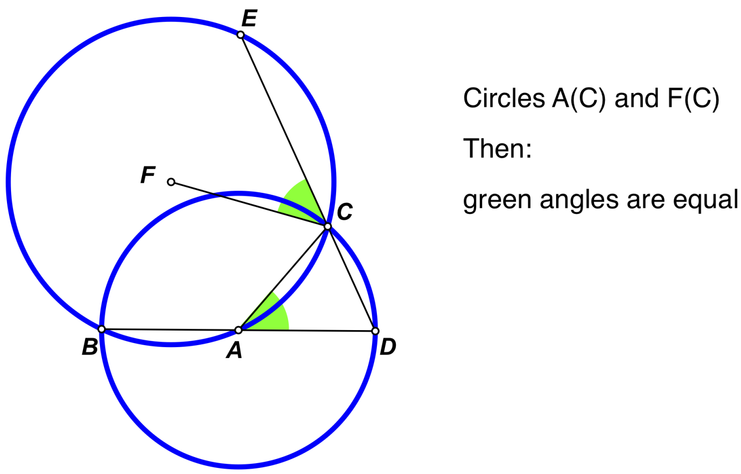
Semicircles are constructed on AB and AC.
D is any point on $\overset{\frown}{AC}$.
Circle (E) is tangent to circle (AB) at F, and passes through C and D.

Then:

green angles are equal

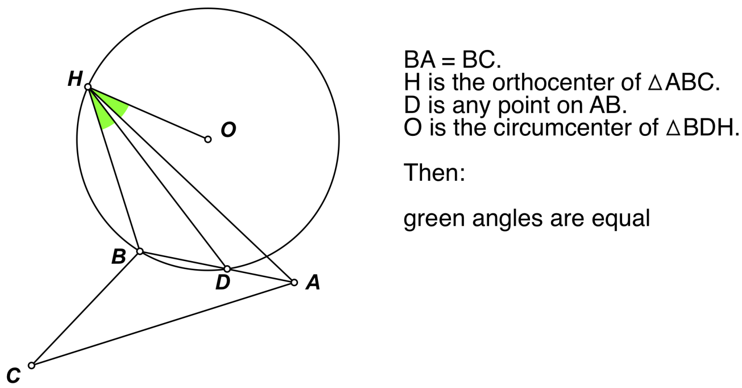
Proof. See [74]. □

Theorem 49.



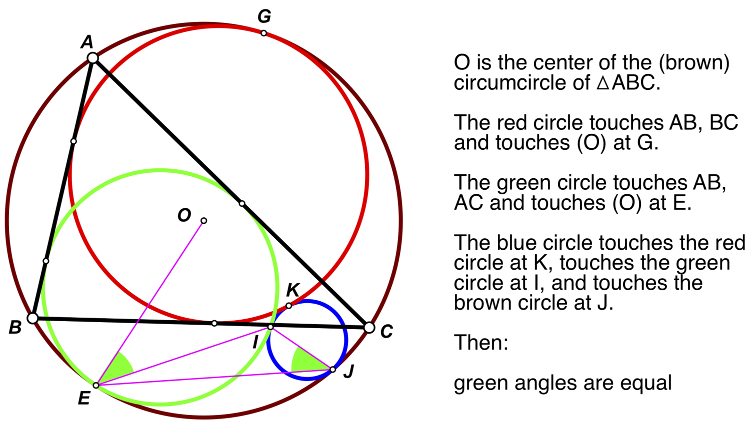
Proof. See [46]. □

Theorem 50.



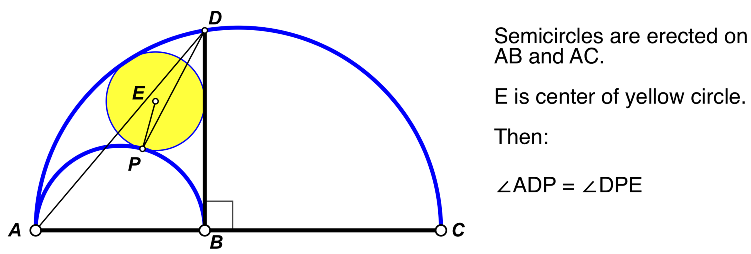
Proof. See [45]. □

Theorem 51.



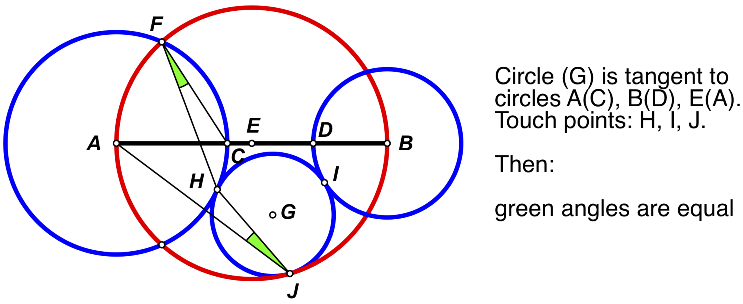
Proof. See [70]. □

Theorem 52.



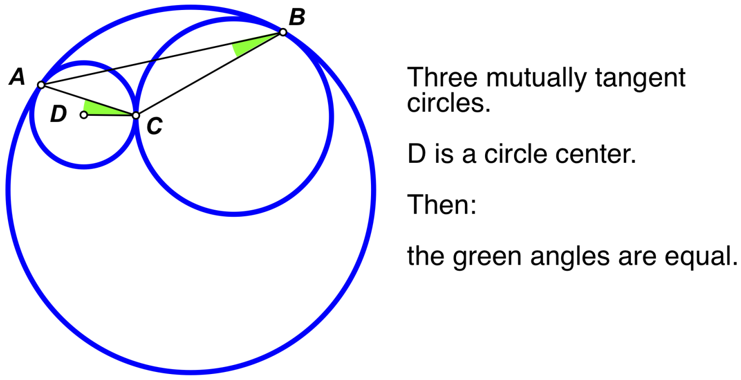
Proof. See [71]. □

Theorem 53.



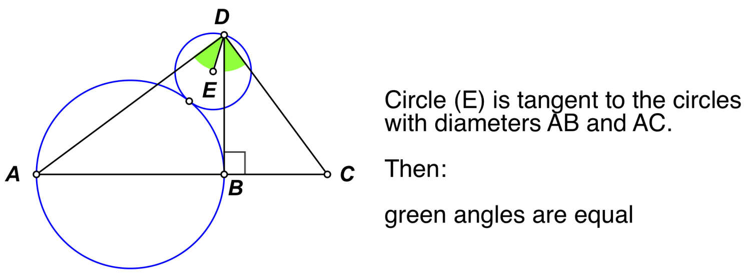
Proof. See [72]. □

Theorem 54.



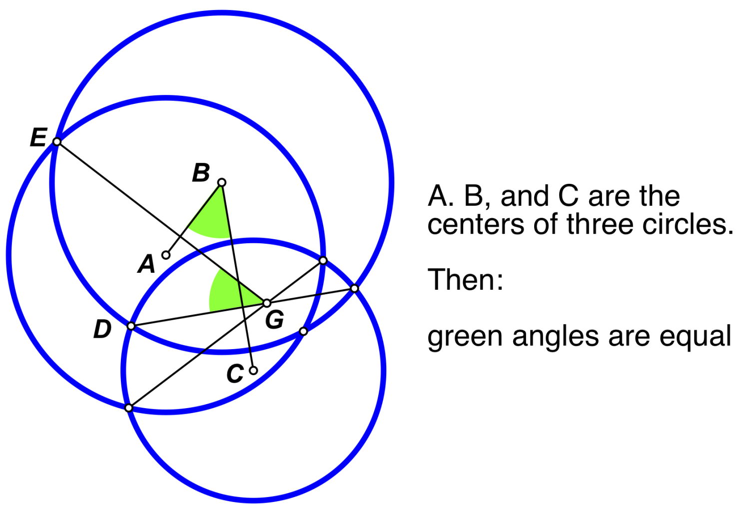
Proof. See [79]. □

Theorem 55.



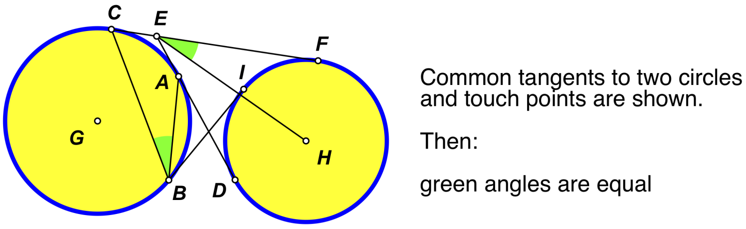
Proof. See [78]. □

Theorem 56.



Proof. See [76]. □

Theorem 57.

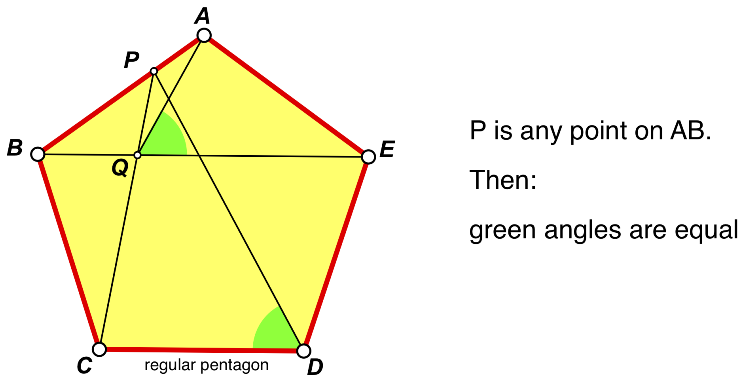


Proof. See [75]. □

6.6. Regular Pentagons.

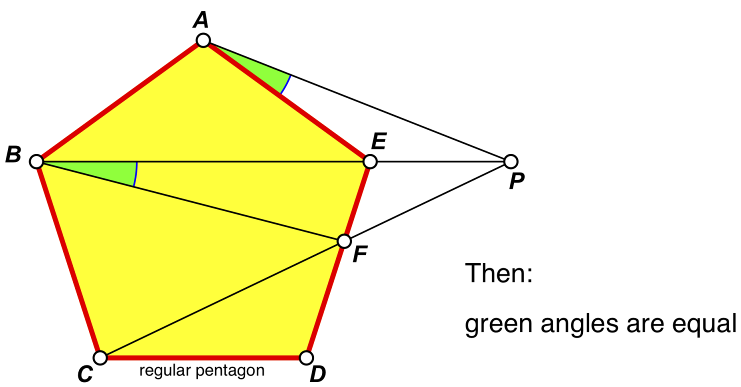
We found the following results involving pairs of equal angles associated with regular pentagons.

Theorem 58.

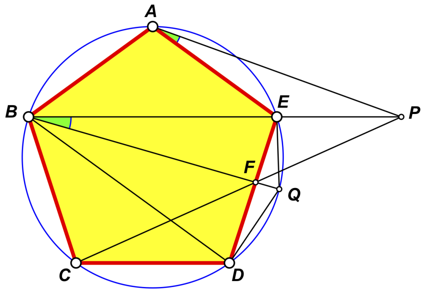


Proof. See [39]. □

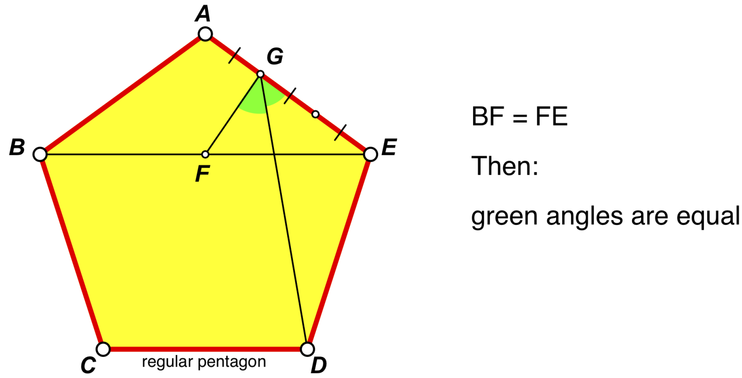
Theorem 59 (Knop [6])



Proof. (Orellana [8])
Let BF meet the circumcircle of the pentagon at Q. $BE = BD \Rightarrow \angle BED = \angle EDB$. $BEQD$ cyclic $\Rightarrow \angle EQB = \angle EDB = \angle BED = \angle BQD$. $\angle EQB = \angle BQD$ means that QF bisects $\angle EQD$. Thus, $EQ/QD = EF/FD$. $CD \parallel BP \Rightarrow \angle EPF = \angle FCD \Rightarrow \triangle EPF \sim \triangle DCF \Rightarrow EF/FD = EP/CD = EP/AE$. $\angle AEB = 36° \Rightarrow \angle PEA = 144° = \angle EQD$. $\angle PEA = \angle EQD$ and $EQ/QD = EF/FD \Rightarrow \triangle PEA \sim \triangle DEQ \Rightarrow \angle EAP = \angle QDE = \angle QBE$. □

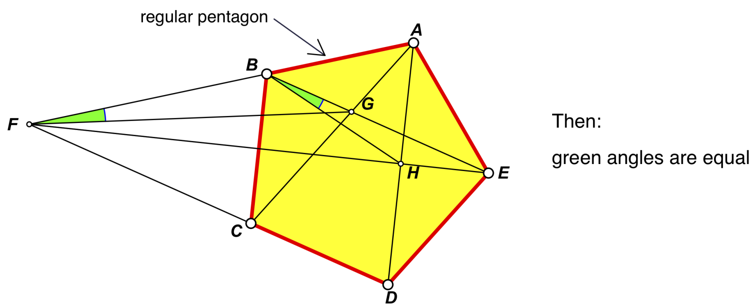


Theorem 60 (Knop)



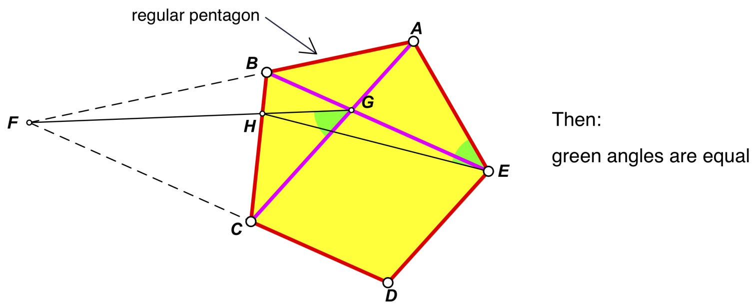
Proof. See [5]. □

Theorem 61.



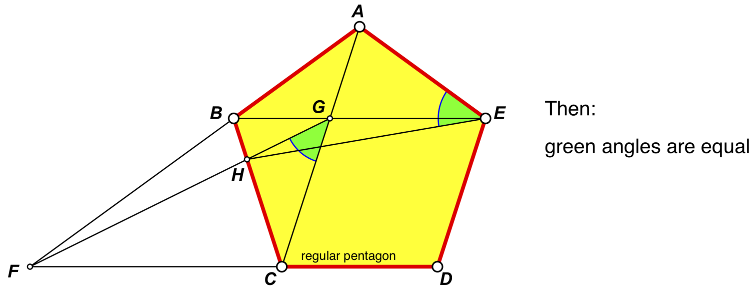
Proof. See [13]. □

Theorem 62.



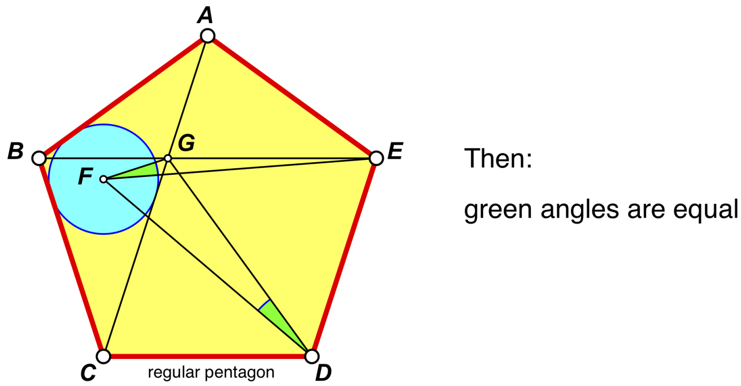
Proof. See [12]. □

Theorem 63.



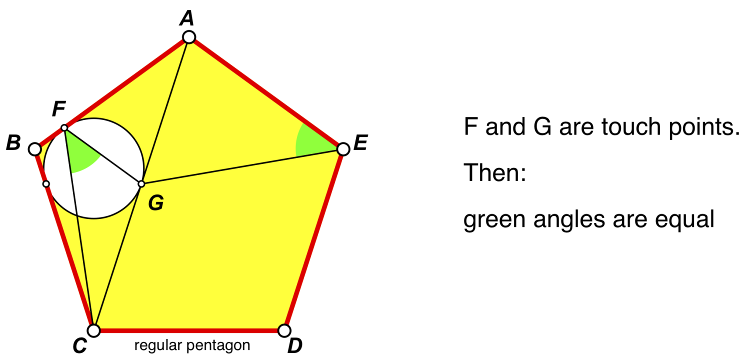
Proof. See [25]. □

Theorem 64.



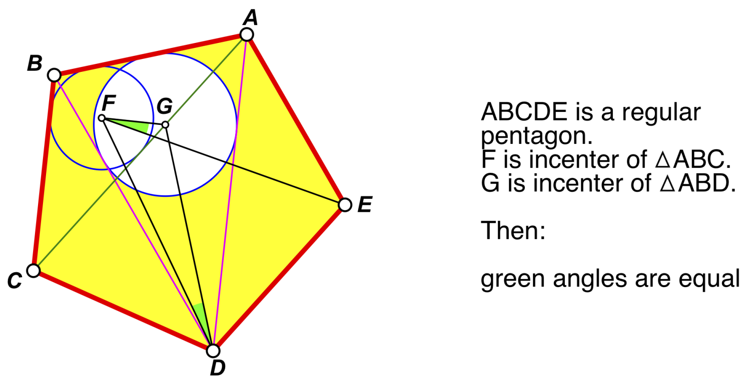
Proof. See [26]. □

Theorem 65.



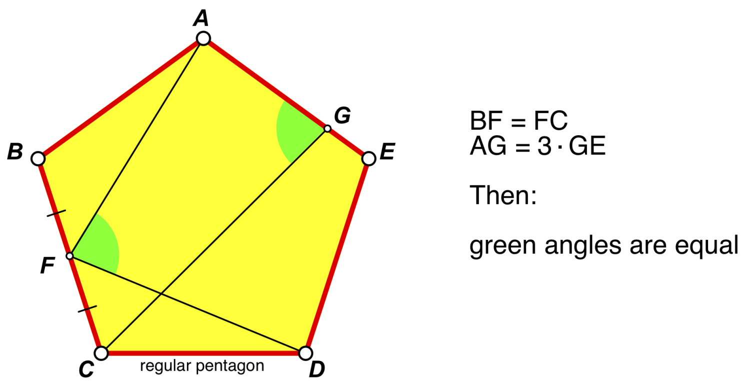
Proof. See [27]. See [2] for a pretty proof. □

Theorem 66.



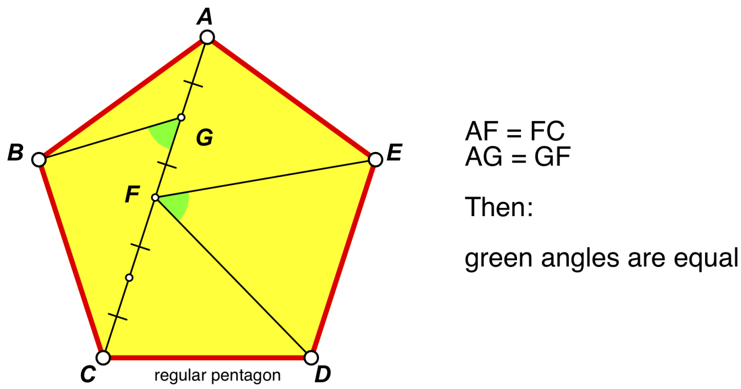
Proof. See [11]. □

Theorem 67.

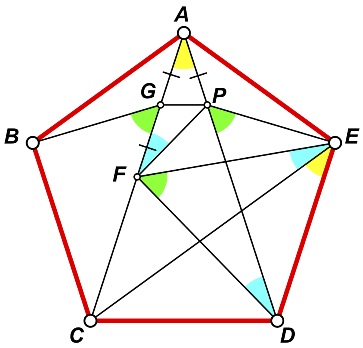


Proof. See [28]. □

Theorem 68.

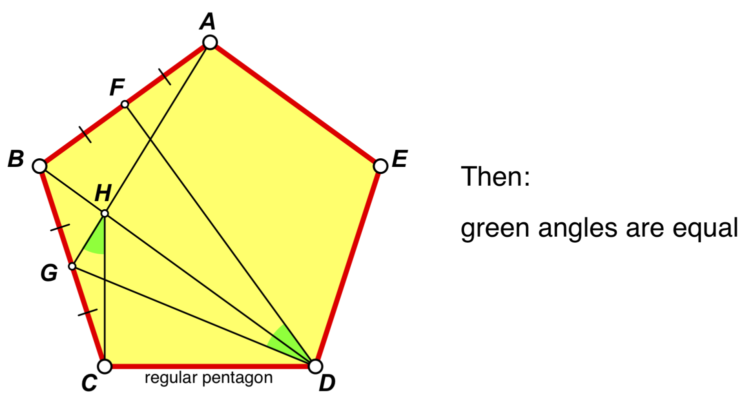


Proof. (Çetin [4])
Draw $GP \parallel CD$ with P on AD so that $AG = AP$. $\triangle BCG \cong \triangle EDP \Rightarrow \angle BGC = \angle DPE$. $\triangle AFP \sim$ADF because $AF/AP = 2 = AD/AF$ and $\angle FAP = \angle DAF$. $AEDC$ cyclic $\Rightarrow \angle DAF = \angle CED$. $\triangle ADF \cong \triangle CEF \Rightarrow \angle CEF = \angle ADF$. $\angle FPD = \angle PFA + \angle FAP = \angle FEC + \angle CED = \angle FED$. $\angle FPD = \angle FED \Rightarrow PEDF$ cyclic $\Rightarrow \angle DFE = \angle DPE = \angle BGC$.



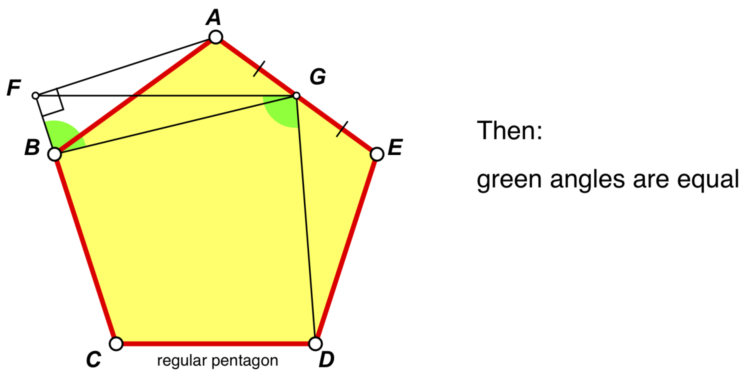
□

Theorem 69.



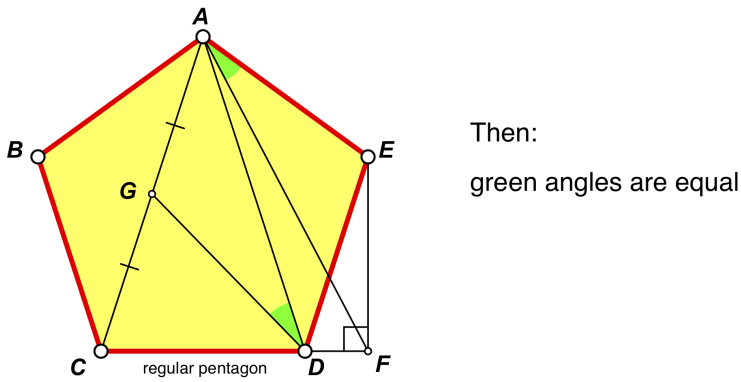
Proof. See [44]. □

Theorem 70.

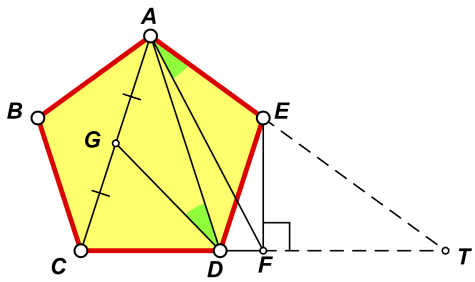


Proof. See [30]. □

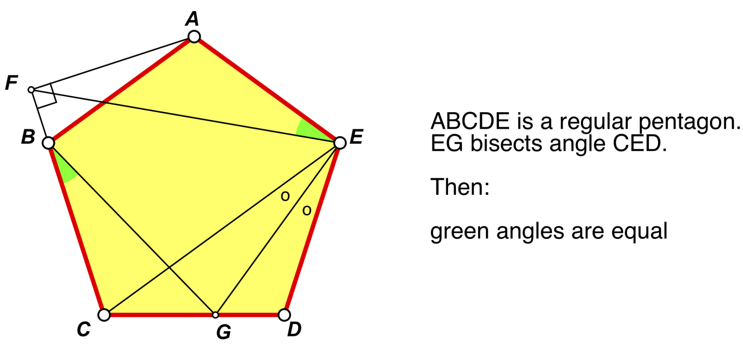
Theorem 71.



Proof. (Khachaturyan [7])
Let $T = AE \cap CD$. $BE \parallel CD$, $BD \parallel AE$, so $BETD$ is a parallelogram. Thus, $TE = BD = CE$. Now $CE = ET$ implies F is the midpoint of CT. The green angles are equal because they are corresponding angles (between side and median) in similar triangles $\triangle ATC$ and $\triangle DAC$. □

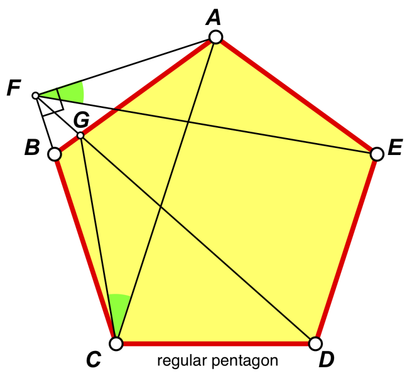


Theorem 72.



Proof. See [31]. □

Theorem 73.

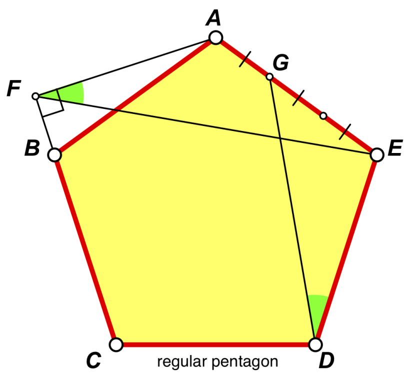


Then:

green angles are equal

Proof. See [32]. □

Theorem 74.

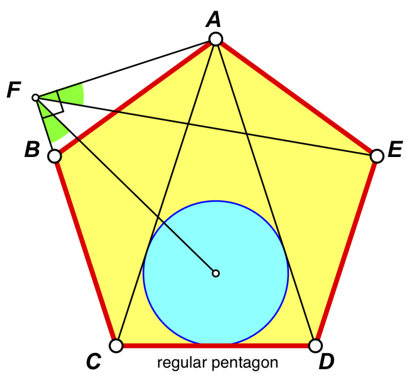


Then:

green angles are equal

Proof. See [33]. □

Theorem 75.

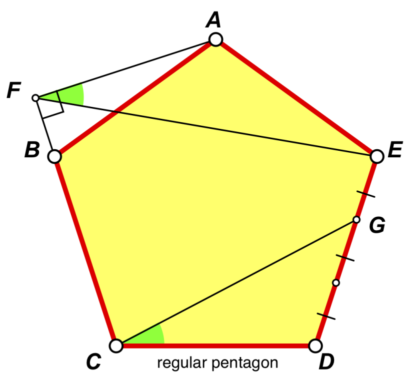


Then:

green angles are equal

Proof. See [34]. □

Theorem 76.

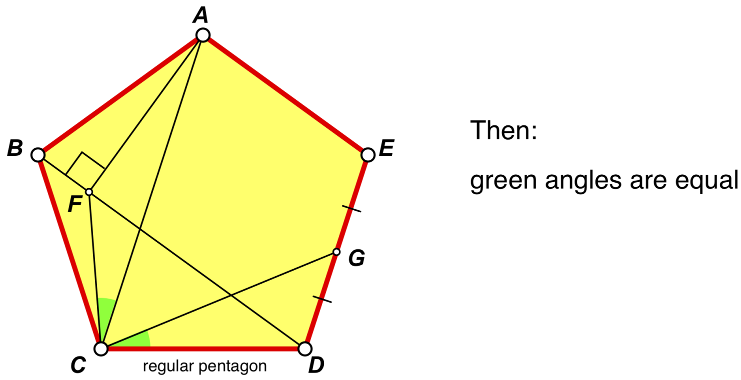


Then:

green angles are equal

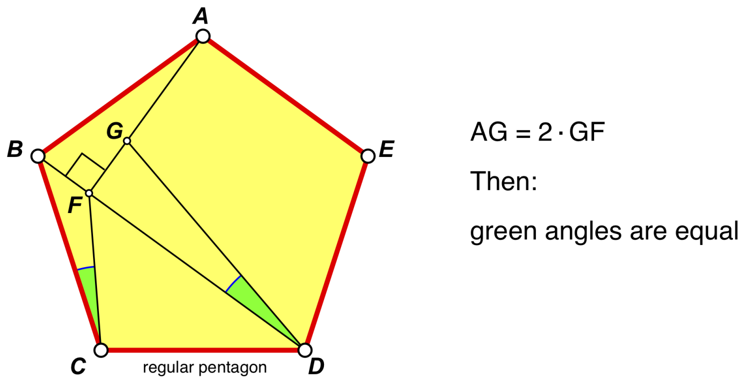
Proof. See [36]. □

Theorem 77.



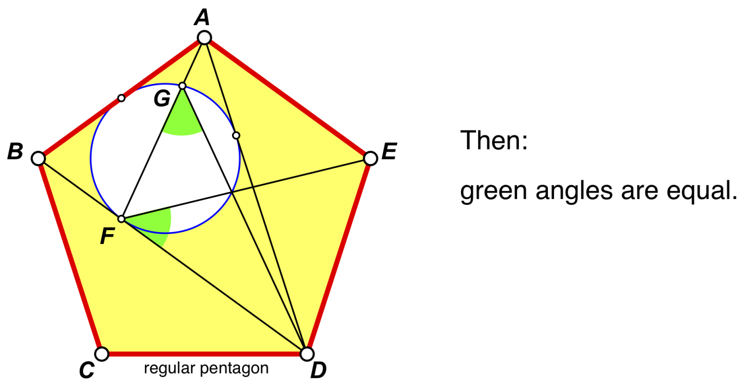
Proof. See [37]. □

Theorem 78.



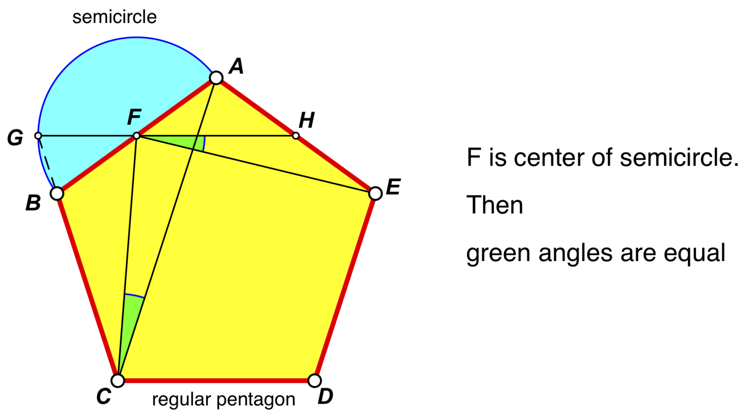
Proof. See [38]. □

Theorem 79.



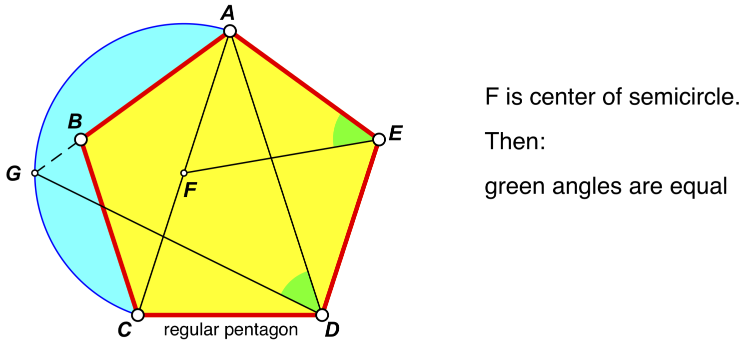
Proof. See [41]. □

Theorem 80.



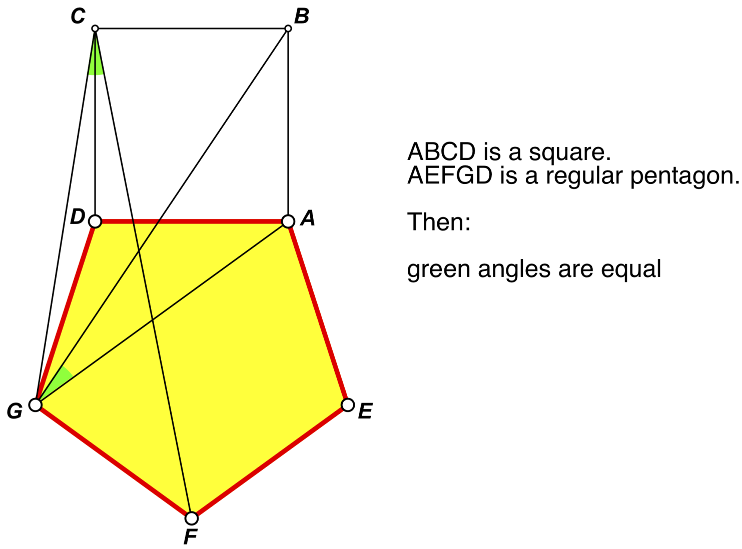
Proof. See [17]. □

Theorem 81.



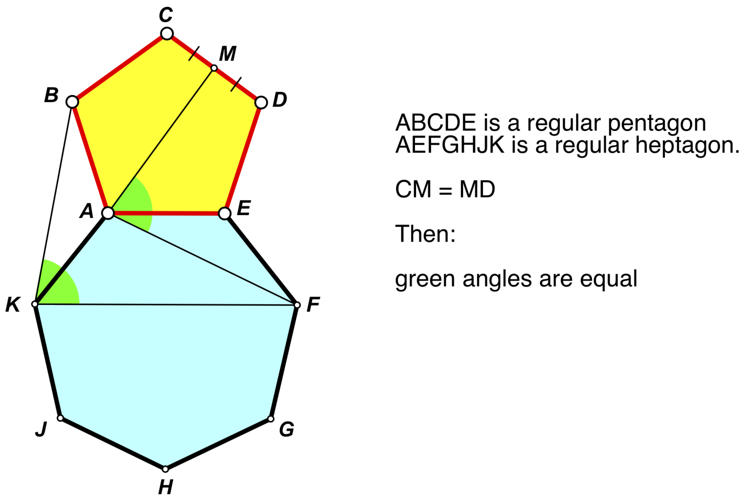
Proof. See [18]. □

Theorem 82.



Proof. See [9]. □

Theorem 83.



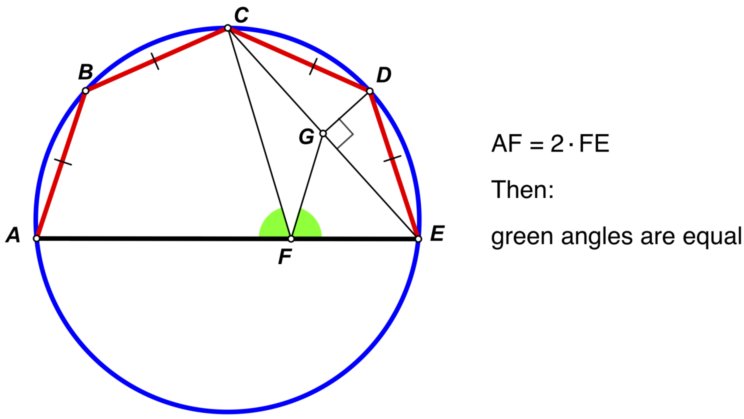
Proof. See [29]. □

6.7. Pseudo-Regular Pentagons.

A *pseudo-regular pentagon* is a cyclic pentagon with four equal sides.

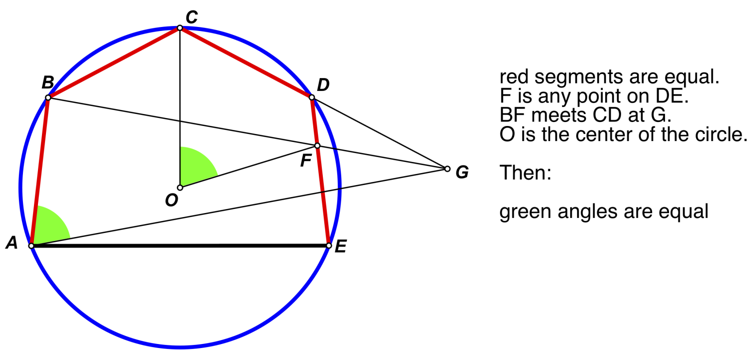
We found the following results involving pairs of equal angles associated with pseudo-regular pentagons.

Theorem 84.



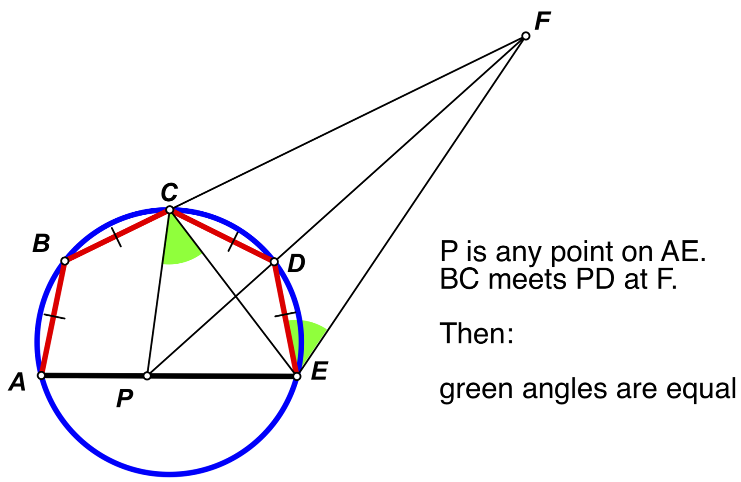
Proof. See [47]. □

Theorem 85.



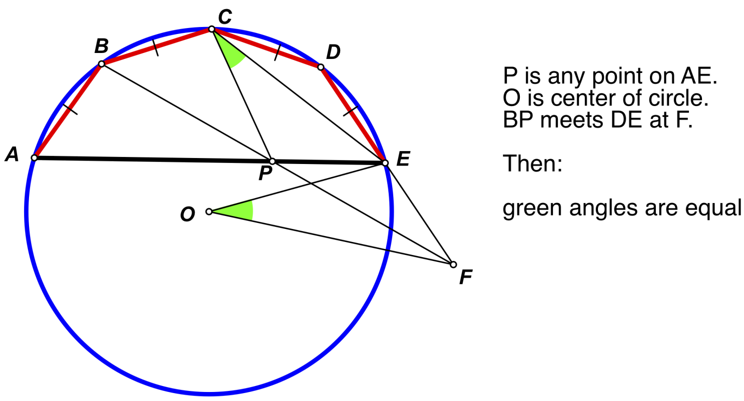
Proof. See [48]. □

Theorem 86.



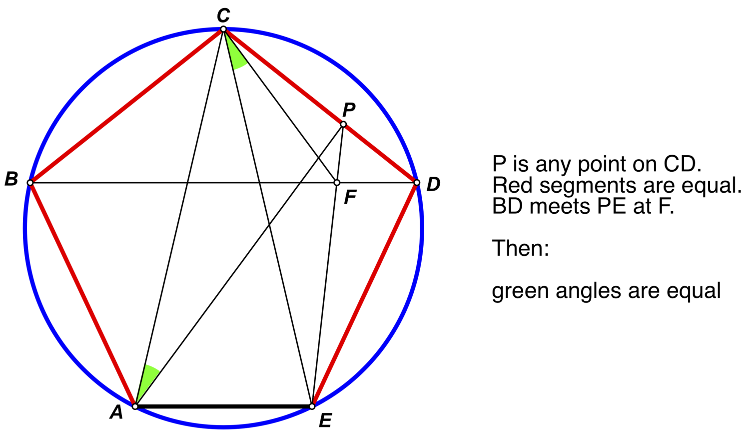
Proof. See [49]. □

Theorem 87.



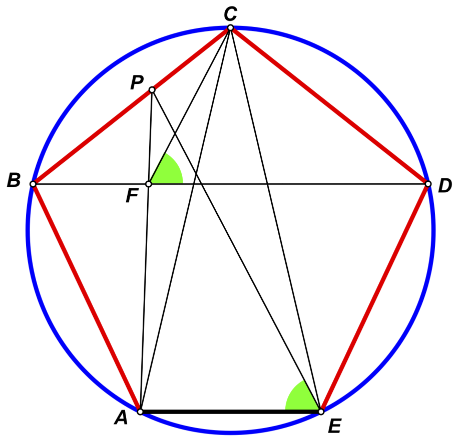
Proof. See [50]. □

Theorem 88.



Proof. See [51]. □

Theorem 89.



P is any point on BC.
Red segments are equal.
BD meets PA at F.

Then:

green angles are equal

Proof. See [40]. □

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