International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 9, 2024, pp.1-8
Received February 5, 2024. Published on-line June 28, 2024
web: http://www.journal-1.eu/
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On a pair of twin conics

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Abstract. We define a pair of conic sections associated to a triangle and one of its Cevian triangles and discuss some of their properties in light of computer verified algebraic identities.

Keywords. conic section, homothetic, triangle geometry, computer-discovered mathematics, Euclidean geometry.

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Setup and statement of theorems

Let A_1, B_1, C_1, P be four points in general position[2] and let $\triangle A_2B_2C_2$ be the Cevian triangle of $\triangle A_1B_1C_1$ with respect to P. If U, V, W are the respective midpoints of B_2C_2, C_2A_2, A_2B_2 then it can be proved, as we shall see, that the three lines A_1U, B_1V, C_1W are concurrent, at Q say. Let $\mathcal{C}_1$ be the conic through A_1, B_1, C_1, P and Q, and let $\mathcal{C}_2$ be the conic through A_2, B_2, C_2, P and Q.

It turns out that regardless of the relative positions of A_1, B_1, C_1 and P, these two conics are very closely related.

Theorem 1.1. *The conic sections $\mathcal{C}_1$ and $\mathcal{C}_2$ defined above are either both ellipses, both parabolas or both hyperbolas. Furthermore,*

(i) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are both ellipses, then they have the same eccentricity and their major axes are parallel.
(ii) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are both parabolas, then their axes of symmetry are parallel.
(iii) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are both hyperbolas, then the asymptotes of one are parallel to those of the other.

Theorem 1.2. *If R_1, S_1 are distinct points on $\mathcal{C}_1$ and R_2, S_2 are distinct points on $\mathcal{C}_2$ such that R_1, P, R_2 are collinear and S_1, Q, S_2 are collinear, then R_1S_1 is parallel to R_2S_2.*

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[2]Specifically, we shall assume throughout that P does not line on the extended sides nor medians of $\triangle A_1B_1C_1$, nor on the line through A_1 parallel to B_1C_1 etc.

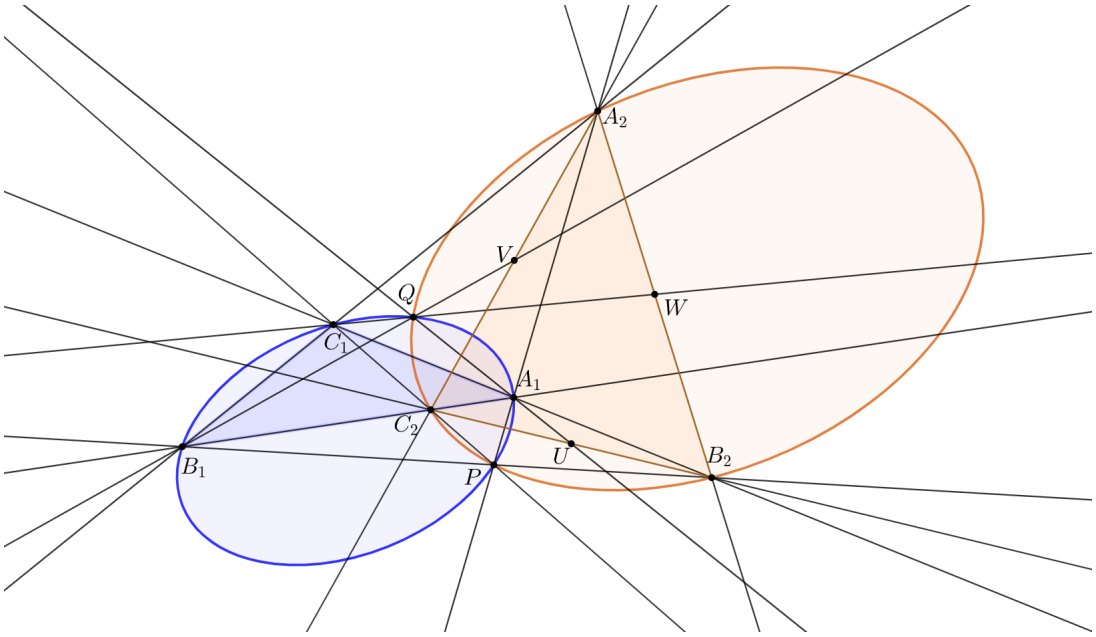

FIGURE 1. Twin ellipses.

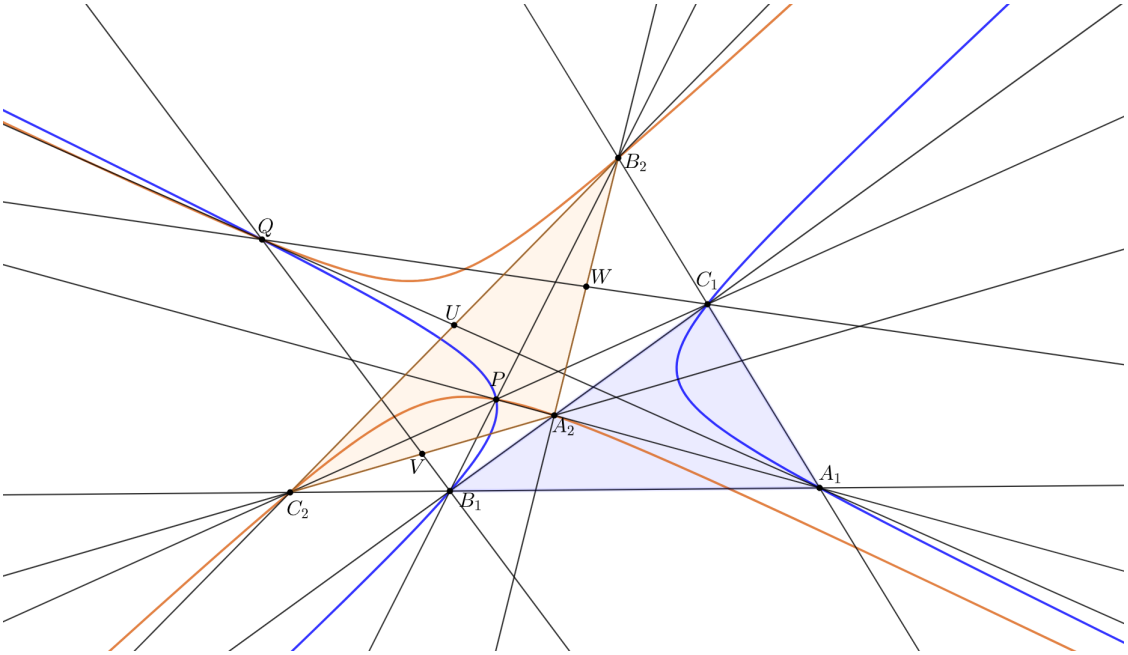

FIGURE 2. Twin hyperbolas.

Remark. *Note that we can drop the condition from Theorem 1.2 that R_1 and S_1 be distinct provided we interpret R_1S_1 to be the tangent to $\mathcal{C}_1$ at R_1 when $R_1 = S_1$ and likewise if $R_2 = S_2$.*

Remark. *Theorem 1.2 is not specific to the particular conics $\mathcal{C}_1$, $\mathcal{C}_2$ but rather is true for any pair of conics that intersect at two points P, Q and satisfy (i-iii) of Theorem 1.1.*

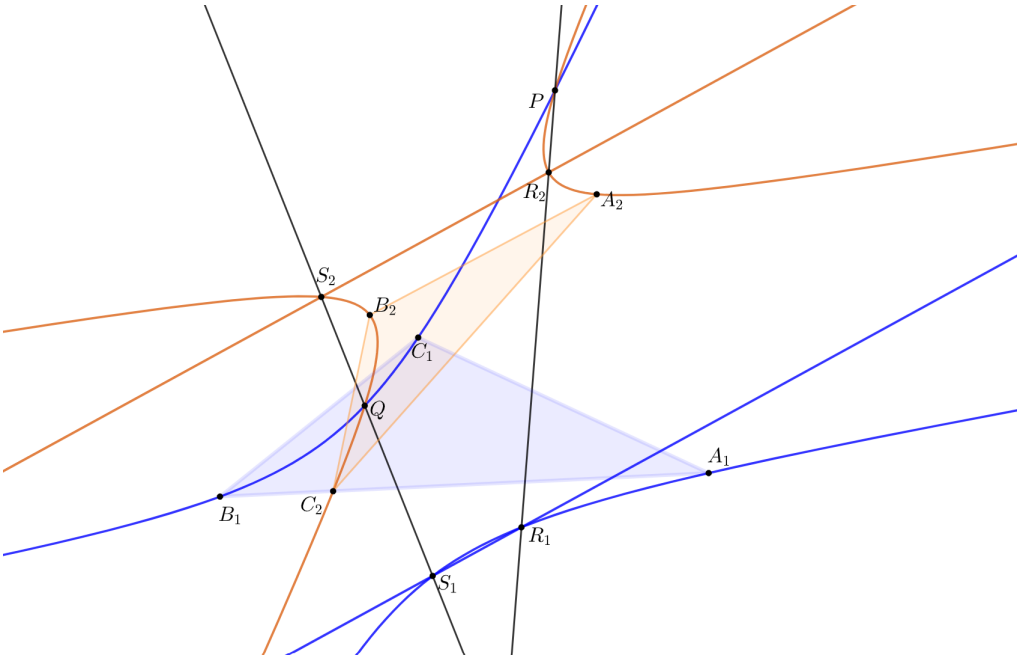

FIGURE 3. Theorem 1.2 parallel chords.

The following Corollary follows directly from Theorem 1.2.

Corollary 1.1. *Let A_1' be the intersection of the line through B_1 parallel to A_2B_2 and the line through C_1 parallel to A_2C_2. Let A_2' be the intersection of the line through B_2 parallel to A_1B_1 and the line through C_2 parallel to A_1C_1. Define B_1', B_2', C_1', C_2' symmetrically. Then*

(i) The points $A_1^{'}, B_1^{'}, C_1^{'}$ lie on $\mathcal{C}_1$.
(ii) The points $A_2^{'}, B_2^{'}, C_2^{'}$ lie on $\mathcal{C}_2$.
(iii) The six lines $A_1^{'}A_2, B_1^{'}B_2, C_1^{'}C_2, A_2^{'}A_1, B_2^{'}B_1, C_2^{'}C_1$ are concurrent at Q.
(iv) The tangent to $\mathcal{C}_1$ at A_1 is parallel to A_2A_2' etc.
(v) The tangent to $\mathcal{C}_2$ at A_2 is parallel to A_1A_1' etc.

Proof. For (i), let X be the point on the line A_2Q other that Q where it meets $\mathcal{C}_1$. By Theorem 1.2 applied with $R_1 = X, S_1 = B_1$ and $R_2 = A_2, S_2 = B_2$ (since B_1, P, B_2 are collinear) we have that X is on the line through B_1 parallel to A_2B_2. By a symmetric argument it is also on the line through C_1 parallel to A_2C_2 and hence X is equal to the point A_1' and this point lies on $\mathcal{C}_1$. By symmetry B_1', C_1' are also on $\mathcal{C}_1$.

The proof of (ii) is similar. This time let X be the point other than Q where A_1Q meets $\mathcal{C}_2$. Apply Theorem 1.2 with $R_1 = A_1, S_1 = B_1$ and $R_2 = X, S_2 = B_2$ to deduce that X lies on the line through B_2 parallel to A_1B_1. By symmetry it also lies on the line through through C_2 parallel to A_1C_1 and is therefore equal to the point A_2' which must therefore lie on $\mathcal{C}_1$. Likewise for B_2', C_2'.

Note that we have proved (iii) in the course of proving (i) and (ii).

Point (iv) now follows from (ii) and Theorem 1.2 together with the facts that A_1, P, A_2 are collinear and A_1, Q, A_2' are collinear etc. Point (v) follows similarly from (i) and the collinearity of A_1, P, A_2 and of A_1', Q, A_2 etc. □

2. Proof of theorem 1.1

Let P have areal coordinates (l, m, n) with respect to $\triangle A_1B_1C_1$. So A_1, B_1, C_1 are represented by $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ and A_2, B_2, C_2 by $(0, m, n), (l, 0, n), (l, m, 0)$. Note these are not normalised so coordinates don't have to sum to 1. Now U is the midpoint of B_2, C_2 and so is given by $(l/(n+l)+l/(l+m), m/(l+m), n/(n+l))$. It can be verified that the point Q with coordinates $(l(m+n), m(n+l), n(l+m))$ lies on A_1U. By symmetry, Q also lies on B_1V and C_1W. For an introduction to areal coordinates and how they are used in proofs of these kind of coincidences, see for example [1].

If we define the two matrices

$$\Phi_1^{\text{areal}} = \begin{pmatrix} 0 & n(l^2-m^2) & m(n^2-l^2) \\ n(l^2-m^2) & 0 & l(m^2-n^2) \\ m(n^2-l^2) & l(m^2-n^2) & 0 \end{pmatrix}$$

$$\Phi_2^{\text{areal}} = \begin{pmatrix} 2mn(m-n) & n^2(l-m) & m^2(n-l) \\ n^2(l-m) & 2nl(n-l) & l^2(m-n) \\ m^2(n-l) & l^2(m-n) & 2lm(l-m) \end{pmatrix}$$

then it can be verified that for $i = 1, 2$ the conics $\mathcal{C}_1, \mathcal{C}_2$ are given by the equations in areal coordinates (X, Y, Z)

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}^{\mathrm{T}} \Phi_i^{\text{areal}} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0.$$

This follows by simply checking that they contain the points A_1, B_1, C_1, P, Q and A_2, B_2, C_2, P, Q respectively. We have

$$\det \Phi_1^{\text{areal}} = 2lmn(n^2-m^2)(l^2-n^2)(m^2-l^2)$$

$$\det \Phi_2^{\text{areal}} = 4l^2m^2n^2(n-m)(l-n)(m-l)$$

$$(2lmn) \det \Phi_1^{\text{areal}} = (l+m)(m+n)(n+l) \det \Phi_2^{\text{areal}} \tag{1}$$

At this point it is convenient to translate into Cartesian coordinates. If we write

$$\mathbf{T} = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ 1 & 1 & 1 \end{pmatrix}$$

where in Cartesian coordinates now $A_1 = (p_1, p_2), B_1 = (q_1, q_2), C_1 = (r_1, r_2)$ then a point with areal coordinates (X, Y, X) and $X + Y + Z = 1$ has Cartesian coordinates (x, y) given by

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \mathbf{T} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

Define for $i = 1, 2$ the matrices

$$\Phi_i^{\text{cart}} = \begin{pmatrix} a_i & h_i & g_i \\ h_i & b_i & f_i \\ g_i & f_i & c_i \end{pmatrix} = (\mathbf{T}^{-1})^{\mathrm{T}}\, \Phi_i^{\text{areal}}\, \mathbf{T}^{-1}$$

so that a_i etc are polynomials in the coordinates l, m, n that depend on $p_1, p_2, q_1, q_2, r_1, r_2$. Then the Cartesian coordinate equations for $\mathcal{C}_i$ are

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}^{\mathrm{T}} \Phi_i^{\text{cart}} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$

which can be written equivalently as

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a_i & h_i \\ h_i & b_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} f_i \\ g_i \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x \\ y \end{pmatrix} + c_i = 0.$$

The key identity that implies the close connection between our pair of conics is

$$\begin{pmatrix} a_1 & h_1 \\ h_1 & b_1 \end{pmatrix} = \begin{pmatrix} a_2 & h_2 \\ h_2 & b_2 \end{pmatrix}. \tag{2}$$

In general this identity is much too complicated to write out in full but it can be verified using symbolic algebra software like the python package sympy[3].

It follows immediately from (2) that the discriminants of the two conics, namely $\Delta_i = -4 \det \left(\begin{smallmatrix} a_i & h_i \\ h_i & b_i \end{smallmatrix}\right)$, are equal to each other. Furthermore, upon calculating this 2×2 determinant there is a great deal of simplification and we are left with the following expression for Δ_i

$$4(l+m)(l+n)(m+n)\left((l+m)(l+n)(m+n) - 8lmn\right)/(\det \mathbf{T})^2. \tag{3}$$

Hence $\mathcal{C}_1$ and $\mathcal{C}_2$ are both hyperbola/ellipse/parabola according to whether (3) is positive/negative/zero. The equation $l + m = 0$ corresponds to the line through A_1 parallel to B_1C_1 etc. Therefore, it can be seen that the zero locus of the discriminant consists of three lines and a cubic curve which altogether divide the plane into 13 regions. (See Figure 4.)

The type of conic is determined by which region the point P lies in. For 6 of these regions they are ellipses and for 7 they are hyperbolas (including inside $\triangle A_1B_1C_1$). The three lines mentioned are degenerate cases we do not consider further. The cubic curve

$$(l+m)(m+n)(n+l) = 8lmn$$

passes through A_1, B_1, C_1 and when P lies on this curve but is not equal to A_1, B_1 or C_1 the conics are both parabolas.

[3]We could simplify by applying a suitable affine transformation first. For example we could translate C to the origin, rotate B to the y-axis and rescale so that $A = (p, q), B = (0, 1)$ and $C = (0, 0)$ for some p, q. This greatly simplifies the identity (2) but it is still a lot of work to verify by hand.

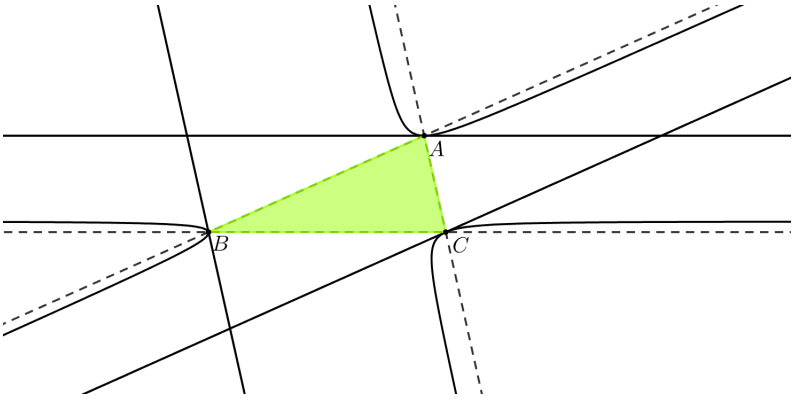

FIGURE 4. Zero locus of conic discriminant (solid line).

Now the eccentricity of $\mathcal{C}_i$ depends only on a_i, b_i, h_i and the sign of $\det \Phi_i^{\text{cart}}$ which is the same as the sign of Φ_i^{areal}. See [2] for an explicit formula for the eccentricity in terms these quantities. By (1) and (2), therefore, they can only have different eccentricities if

$$lmn(l+m)(m+n)(n+l) < 0.$$

And by (3), this can only happen if the discriminants are positive, that is, they are hyperbolas. In particular, if they are ellipses, they have the same eccentricity.

To finish the proof of Theorem 1.1, it remains to prove points $(i), (ii), (iii)$. But these facts also follow from the identity (2) and standard facts about conics. For let us call the *orientation* of the conic the directions of major/minor axes in the case of ellipse, the direction of the axis of symmetry in the case of parabola and the directions of the asymptotes in the case of hyperbola. Then the *orientation* is determined by the matrix $\left(\begin{smallmatrix} a_i & h_i \\ h_i & b_i \end{smallmatrix}\right)$ and is independent of the coefficients f_i, g_i, c_i. See [3],[4],[4] for explicit formulae for the major/minor axes of an ellipse, asymptotes of a hyperbola and axis of symmetry in terms of a_i, b_i, h_i.

3. PROOF OF THEOREM 1.2

Suppose R_1, R_2, S_1, S_2 are as described in Theorem 1.2 so R_1, S_1 lie on $\mathcal{C}_1$ and R_2, S_2 lie on $\mathcal{C}_2$ with R_1, P, R_2 collinear and S_1, Q, S_2 collinear. The task is to show that the chord R_1S_1 is parallel to the chord R_2S_2. We shall consider separately the three cases in which our pair of conics are ellipses, parabolas or hyperbolas.

For each, we make use of the fact that affine transformations preserve parallelism to first transform our conics into standard forms that are easily parameterised. We can do this because Theorem 1.1. will ensure that the transformation taking $\mathcal{C}_1$ to a standard form also takes $\mathcal{C}_2$ to a corresponding standard form. Since we may assume without loss of generality that the conics are in the standard form (to be described) we will use the same labels for the transformed points.

3.1. **Ellipses.** Suppose first that they are both ellipses. By an affine transformation we can map $\mathcal{C}_1$ to the circle with equation $x^2 + y^2 = 1$. Because $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same eccentricity and their major axes are parallel, the same map will transform $\mathcal{C}_2$ into some other circle, with equation $(x-a)^2 + (y-b)^2 = c^2$ for some a, b, c, say. In this case the result is an elementary exercise in angle chasing

using basic circle theorems. In particular,

$$\angle PR_1S_1 = \pi - \angle S_1QP = \angle PQS_2 = \pi - \angle S_2R_2P$$

and hence R_1S_1 is parallel to R_2S_2 as required.

3.2. **Hyperbolas.** Suppose now that they are both hyperbolas. By an affine transformation we can map $\mathcal{C}_1$ to the hyperbola with equation $xy = 1$. By Theorem 1.1, this map will transform $\mathcal{C}_1$ into some other hyperbola with parallel asymptotes. That is, to a hyperbola with equation $(x-a)(y-b) = c$ for some a, b and $c \neq 0$. Now we can easily describe the Cartesian coordinates of the (transformed) points $R_1, S_1 \in \mathcal{C}_1$ and $R_2, S_2 \in \mathcal{C}_2$ with parameters r_1, r_2, s_1, s_2 as follows:

$$R_1 = (r_1, 1/r_1), \quad S_1 = (s_1, 1/s_1), \quad R_2 = (r_2{+}a, c/r_2{+}b), \quad S_2 = (s_2{+}a, c/s_2{+}b).$$

Since P, Q lie on both conics we can write for some p_1, p_2, q_1, q_2

$$P = (p_1, 1/p_1) = (p_2 + a, c/p_2 + b), \quad Q = (q_1, 1/q_1) = (q_2 + a, c/q_2 + b).$$

It is now a matter of combing these relations with the fact that R_1, P, R_2 are collinear and S_1, Q, S_2 are collinear to deduce the result. The collinearity of R_1, P, R_2 implies that

$$(p_1 - r_1)((c/p_2 + b) - (c/r_2 + b)) = ((p_2 + a) - (r_2 + a))(1/p_1 - 1/r_1)$$

which, since R_1, R_2 are distinct, is equivalently to

$$cp_1r_1 = p_2r_2.$$

It follows likewise from the collinearity of S_1, P, S_2 that $cq_1s_1 = q_2s_2$. To prove that R_1S_1 is parallel to R_2S_2, we need to establish the relation

$$(r_1 - s_1)((c/r_2 + b) - (c/s_2 + b)) = ((r_2 + a) - (s_2 + a))(1/r_1 - 1/s_1)$$

that is

$$cr_1s_1 = r_1s_2.$$

Using $cp_1r_1 = p_2r_2$ and $cq_1s_1 = q_2s_2$, it suffices to show that $cp_1q_1 = p_2q_2$. Or, after eliminating p_2, q_2, to show that $cp_1q_1 = (p_1 - a)(q_1 - a)$, which is convenient to write as

$$(c-1)p_1q_1 + a(p_1 + q_1) - a^2 = 0.$$

Using that fact that p_1, q_1 are the two distinct roots of the equation $1/X = c/(X-a)-b$, or $bX^2+(1+ab-c)X+a = 0$, we must have $p_1+q_1 = -(1+ab-c)/b$ and $p_1q_1 = a/b$. With these expressions it therefore only remains to verify the identity

$$(c-1)a/b + a(1 + ab - c)/b - a^2 = 0.$$

This completes the proof in the case that $\mathcal{C}_1$ and $\mathcal{C}_2$ are hyperbolas.

3.3. **Parabolas.** The case in which they are parabolas is very similar except we have a different standard form and different parametrisation. This time we map $\mathcal{C}_1$ to the curve defined by the equation $y = x^2$ and $\mathcal{C}_2$ is transformed under the same map to a parabola with the same axis of symmetry, the y-axis. That is, to $y = c(x-a)^2 + b$ for some a, b, c. We must have $c \neq 1$ because we know there are two distinct points of intersection P and Q. We can this time describe our (transformed) points with the following coordinates

$$R_1 = (r_1, r_1^2), \quad S_1 = (s_1, s_1^2), \quad R_2 = (r_2 + a, cr_2^2 + b), \quad S_2 = (s_2 + a, cs_2^2 + b)$$
$$P = (p_1, p_1^2) = (p_2 + a, cp_2^2 + b), \quad Q = (q_1, q_1^2) = (q_2 + a, cq_2^2 + b).$$

The condition that R_1, P, R_2 be collinear this time is

$$(p_1 - r_1)((cp_2^2 + b) - (cr_2^2 + b)) = ((p_2 + a) - (r_2 + a))(p_1^2 - r_1^2)$$

or, since R_1, R_2 are distinct,

$$c(p_2 + r_2) = p_1 + r_1.$$

From the collinearity of S_1, Q, S_2 we likewise have $c(q_2 + s_2) = q_1 + s_1$. To prove that R_1S_1 is parallel to R_2S_2, we need to establish the relation

$$(r_1 - s_1)(cr_2^2 - cs_2^2) = (r_1^2 - s_1^2)(r_2 - s_2)$$

that is

$$c(r_2 + s_2) = r_1 + s_1.$$

Using $c(p_2 + r_1) = p_1 + r_1$ and $c(q_2 + s_2) = q_1 + s_1$, it therefore suffices to show that $c(p_2 + q_2) = p_1 + q_1$. Or, after eliminating p_2, q_2, to show that

$$(c-1)(p_1 + q_1) - 2ac = 0$$

Using that fact that p_1, q_1 are the two distinct roots of the equation $X^2 = c(X - a)^2 - b$, or $(c-1)X^2 - 2acX + a^2c - b = 0$ we must have $p_1 + q_1 = 2ac/(c-1)$ as required. This completes the proof in the case that $\mathcal{C}_1$ and $\mathcal{C}_2$ are parabolas and the proof of Theorem 1.2.

4. FINAL THOUGHTS

Our proof does not really explain why the crucial identity (2) holds. We leave the task providing a proper explanation to an interested reader to think about.

One wonders for example what further insight might be gained if it could be shown more directly that P and Q both lie on the circumcircle of $\triangle A_1B_1C_1$ if and only if they both lie on the circumcircle of $\triangle A_2B_2C_2$.

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