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Generalization of the Jha-Savaran Theorem

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1. Introduction

The Jha-Savaran theorem [1] is a recent result which generalizes the Napoleon theorem [2] that the centroids of the equilateral triangles constructed on the sides a triangle form an equilateral triangle. Here, we give a generalization of the former result using the centroid generalization discussed in [3].

2. Results

Theorem 2.1. *Let $A, B, C, D, E,$ and F be the vertices of a given hexagon.*
Let $A, B,$ and G form an equilateral triangle on the hexagon.
Let $B, C,$ and H form an equilateral triangle on the hexagon.
Let $C, D,$ and I form an equilateral triangle on the hexagon.
Let $D, E,$ and J form an equilateral triangle on the hexagon.
Let $E, F,$ and K form an equilateral triangle on the hexagon.
Let $A, F,$ and L form an equilateral triangle on the hexagon.
Let M be some point in the plane.
Let N be a point on line GM.
Let O be the point on line HM such that $HO : MO = GN : MN$.
Let P be the point on line IM such that $IP : MP = GN : MN$.
Let Q be the midpoint of N and O.
Let R be the midpoint of N and P.
Let S be the midpoint of O and P.
Let lines $GS, HR,$ and IQ intersect at point T.

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Let U be the point on line JM such that $JU : MU = GN : MN$.
Let V be the point on line KM such that $KV : MV = GN : MN$.
Let W be the midpoint of P and U.
Let Z be the midpoint of P and V.
Let A_1 be the midpoint of U and V.
Let lines IA_1, JZ, and KW intersect at point B_1.
Let C_1 be the point on line LM such that $LC_1 : MC_1 = GN : MN$.
Let D_1 be the midpoint of N and V.
Let E_1 be the midpoint of N and C_1.
Let F_1 be the midpoint of V and C_1.
Let lines GF_1, KE_1, and LD_1 intersect at point G_1.
Then T, B_1, and G_1 form an equilateral triangle.

Proof. Without loss of generality, we will work in Cartesian coordinates.
We split the problem into two cases: where the triangles are external to the hexagon and where they are internal to the hexagon.
Case 1: the triangles are all external to the hexagon.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates (x_E, y_E).
F has coordinates (x_F, y_F).
G has coordinates $(1/2, -3^{1/2}/2)$.
H has coordinates $(x_C/2 + (3^{1/2}y_C)/2 + 1/2, y_C/2 - (3^{1/2}(x_C - 1))/2)$.
I has coordinates
$(x_C/2 + x_D/2 - (3^{1/2}y_C)/2 + (3^{1/2}y_D)/2, y_C/2 + y_D/2 + (3^{1/2}(x_C - x_D))/2)$.
J has coordinates
$(x_E/2 + x_D/2 + (3^{1/2}y_E)/2 - (3^{1/2}y_D)/2, y_E/2 + y_D/2 - (3^{1/2}(x_E - x_D))/2)$.
K has coordinates
$(x_E/2 + x_F/2 - (3^{1/2}y_E)/2 + (3^{1/2}y_F)/2, y_E/2 + y_F/2 + (3^{1/2}(x_E - x_F))/2)$.
L has coordinates
$(x_F/2 - (3^{1/2}y_F)/2, y_F/2 + (3^{1/2}x_F)/2)$.
M has coordinates (x_M, y_M).
N has coordinates $(x_N, (y_M(2x_N - 1) - 3^{1/2}(x_M - x_N))/(2x_M - 1))$.
O has coordinates
$((x_Cx_M - x_N(x_C - 2x_M + 1) + 3^{1/2}y_C(x_M - x_N))/(2x_M - 1), (y_M(2x_N - 1) + (y_C - 3^{1/2}(x_C - 1))(x_M - x_N))/(2x_M - 1))$.
P has coordinates
$((x_M(x_C + x_D - 1) - x_N(x_C + x_D - 2x_M) - 3^{1/2}y_C(x_M - x_N) + 3^{1/2}y_D(x_M - x_N))/(2x_M - 1), (y_M(2x_N - 1) + (x_M - x_N)(y_C + y_D + 3^{1/2}(x_C - x_D)))/(2x_M - 1))$.
Q has coordinates
$(x_N/2 + (x_Cx_M - x_N(x_C - 2x_M + 1) + 3^{1/2}y_C(x_M - x_N))/(2(2x_M - 1)), (y_M(4x_N - 2) + (y_C - 3^{1/2}x_C)(x_M - x_N))/(4x_M - 2))$.
R has coordinates
$(x_N/2 + (x_M(x_C + x_D - 1) - x_N(x_C + x_D - 2x_M) - 3^{1/2}y_C(x_M - x_N) + 3^{1/2}y_D(x_M - x_N))/(2(2x_M - 1)), (y_M(4x_N - 2) + (x_M - x_N)(y_C + y_D - 3^{1/2}(x_D - x_C + 1)))/(4x_M - 2))$.
S has coordinates

$((x_M(2x_C + x_D - 1) - x_N(2x_C + x_D - 4x_M + 1) + 3^{1/2}y_D(x_M - x_N))/(4x_M - 2), (y_M(4x_N - 2) + (x_M - x_N)(2y_C + y_D - 3^{1/2}(x_D - 1)))/(4x_M - 2))$.
T has coordinates
$(-(x_M(2x_C + x_D) - x_N(2x_C + x_D - 4x_M + 2) + 3^{1/2}y_D(x_M - x_N))/(2x_N - 6x_M + 2), -(y_M(4x_N - 2) + (x_M - x_N)(2y_C + y_D - 3^{1/2}x_D))/(2x_N - 6x_M + 2))$.
U has coordinates
$((x_M(x_D + x_E - 1) - x_N(x_D + x_E - 2x_M) - 3^{1/2}y_D(x_M - x_N) + 3^{1/2}y_E(x_M - x_N))/(2x_M - 1), (y_M(2x_N - 1) + (x_M - x_N)(y_D + y_E + 3^{1/2}(x_D - x_E)))/(2x_M - 1))$.
V has coordinates
$((x_M(x_E + x_F - 1) - x_N(x_E + x_F - 2x_M) - 3^{1/2}y_E(x_M - x_N) + 3^{1/2}y_F(x_M - x_N))/(2x_M - 1), (y_M(2x_N - 1) + (x_M - x_N)(y_E + y_F + 3^{1/2}(x_E - x_F)))/(2x_M - 1))$.
W has coordinates
$(-(x_N(x_C + 2x_D + x_E - 4x_M) - x_M(x_C + 2x_D + x_E - 2) + 3^{1/2}y_C(x_M - x_N) - 3^{1/2}y_E(x_M - x_N))/(4x_M - 2), (y_M(4x_N - 2) + (x_M - x_N)(y_C + 2y_D + y_E + 3^{1/2}(x_C - x_E)))/(4x_M - 2))$.
Z has coordinates
$((x_M(x_C + x_D - 1) + x_M(x_E + x_F - 1) - x_N(x_C + x_D - 2x_M) - x_N(x_E + x_F - 2x_M) - 3^{1/2}y_C(x_M - x_N) + 3^{1/2}y_D(x_M - x_N) - 3^{1/2}y_E(x_M - x_N) + 3^{1/2}y_F(x_M - x_N))/(4x_M - 2), (y_M(4x_N - 2) + (x_M - x_N)(y_C + y_D + y_E + y_F + 3^{1/2}(x_C - x_D + x_E - x_F)))/(4x_M - 2))$.
A_1 has coordinates
$(-(x_N(x_D + 2x_E + x_F - 4x_M) - x_M(x_D + 2x_E + x_F - 2) + 3^{1/2}y_D(x_M - x_N) - 3^{1/2}y_F(x_M - x_N))/(4x_M - 2), (y_M(4x_N - 2) + (x_M - x_N)(y_D + 2y_E + y_F + 3^{1/2}(x_D - x_F)))/(4x_M - 2))$.
B_1 has coordinates
$((x_N(x_C + 2x_D + 2x_E + x_F - 4x_M) - x_M(x_C + 2x_D + 2x_E + x_F - 2) + 3^{1/2}y_C(x_M - x_N) - 3^{1/2}y_F(x_M - x_N))/(2x_N - 6x_M + 2), -(y_M(4x_N - 2) + (x_M - x_N)(y_C + 2y_D + 2y_E + y_F + 3^{1/2}(x_C - x_F)))/(2x_N - 6x_M + 2))$.
C_1 has coordinates
$(-(x_N(x_F - 2x_M) - x_M(x_F - 1) + 3^{1/2}y_F(x_M - x_N))/(2x_M - 1), (y_M(2x_N - 1) + (y_F + 3^{1/2}x_F)(x_M - x_N))/(2x_M - 1))$.
D_1 has coordinates
$((1/2)(x_N + (x_M(x_E + x_F - 1) - x_N(x_E + x_F - 2x_M) - 3^{1/2}y_E(x_M - x_N) + 3^{1/2}y_F(x_M - x_N))/(2x_M - 1)), (y_M(4x_N - 2) + (x_M - x_N)(y_E + y_F - 3^{1/2}(x_F - x_E + 1)))/(4x_M - 2))$.
E_1 has coordinates
$((1/2)(x_N - (x_N(x_F - 2x_M) - x_M(x_F - 1) + 3^{1/2}y_F(x_M - x_N))/(2x_M - 1)), (y_M(4x_N - 2) + (y_F + 3^{1/2}(x_F - 1))(x_M - x_N))/(4x_M - 2))$.
F_1 has coordinates
$(-(x_N(x_E + 2x_F - 4x_M) - x_M(x_E + 2x_F - 2) + 3^{1/2}y_E(x_M - x_N))/(4x_M - 2), (y_M(4x_N - 2) + (x_M - x_N)(y_E + 2y_F + 3^{1/2}x_E))/(4x_M - 2))$.
G_1 has coordinates
$((x_N(x_E + 2x_F - 4x_M + 1) - x_M(x_E + 2x_F - 1) + 3^{1/2}y_E(x_M - x_N))/(2x_N - 6x_M + 2), -(y_M(4x_N - 2) + (x_M - x_N)(y_E + 2y_F + 3^{1/2}(x_E - 1)))/(2x_N - 6x_M + 2))$.
We verify that segments TB_1 and TG_1 are equal in length since we calculate
$(x_T - x_{B_1})^2 + (y_T - y_{B_1})^2 - ((x_T - x_{G_1})^2 + (y_T - y_{G_1})^2)$
$= ((-(x_M(2x_C + x_D) - x_N(2x_C + x_D - 4x_M + 2) + 3^{1/2}y_D(x_M - x_N))/(2x_N - 6x_M + 2)) - ((x_N(x_C + 2x_D + 2x_E + x_F - 4x_M) - x_M(x_C + 2x_D + 2x_E + x_F - 2) + 3^{1/2}y_C(x_M - x_N) - 3^{1/2}y_F(x_M - x_N))/(2x_N - 6x_M + 2)))^2 + ((-(y_M(4x_N - 2) + (x_M - x_N)(2y_C + y_D - 3^{1/2}x_D))/(2x_N - 6x_M + 2)) - (-(y_M(4x_N - 2) + (x_M - x_N)(y_C + 2y_D + 2y_E + y_F + 3^{1/2}(x_C - x_F)))/(2x_N - 6x_M + 2)))^2 - (((-(x_M(2x_C +$

$x_D) - x_N(2x_C + x_D - 4x_M + 2) + 3^{1/2}y_D(x_M - x_N))/(2x_N - 6x_M + 2)) - ((x_N(x_E + 2x_F - 4x_M + 1) - x_M(x_E + 2x_F - 1) + 3^{1/2}y_E(x_M - x_N))/(2x_N - 6x_M + 2)))^2 + ((-(y_M(4x_N - 2) + (x_M - x_N)(2y_C + y_D - 3^{1/2}x_D))/(2x_N - 6x_M + 2)) - (-(y_M(4x_N - 2) + (x_M - x_N)(y_E + 2y_F + 3^{1/2}(x_E - 1)))/(2x_N - 6x_M + 2)))^2) = 0$
and we verify that segments TB_1 and B_1G_1 are equal in length since we calculate
$(x_T - x_{B_1})^2 + (y_T - y_{B_1})^2 - ((x_{B_1} - x_{G_1})^2 + (y_{B_1} - y_{G_1})^2)$
$= ((-(x_M(2x_C + x_D) - x_N(2x_C + x_D - 4x_M + 2) + 3^{1/2}y_D(x_M - x_N))/(2x_N - 6x_M + 2)) - ((x_N(x_C + 2x_D + 2x_E + x_F - 4x_M) - x_M(x_C + 2x_D + 2x_E + x_F - 2) + 3^{1/2}y_C(x_M - x_N) - 3^{1/2}y_F(x_M - x_N))/(2x_N - 6x_M + 2)))^2 + ((-(y_M(4x_N - 2) + (x_M - x_N)(2y_C + y_D - 3^{1/2}x_D))/(2x_N - 6x_M + 2)) - (-(y_M(4x_N - 2) + (x_M - x_N)(y_C + 2y_D + 2y_E + y_F + 3^{1/2}(x_C - x_F)))/(2x_N - 6x_M + 2)))^2 - ((((x_N(x_C + 2x_D + 2x_E + x_F - 4x_M) - x_M(x_C + 2x_D + 2x_E + x_F - 2) + 3^{1/2}y_C(x_M - x_N) - 3^{1/2}y_F(x_M - x_N))/(2x_N - 6x_M + 2)) - ((x_N(x_E + 2x_F - 4x_M + 1) - x_M(x_E + 2x_F - 1) + 3^{1/2}y_E(x_M - x_N))/(2x_N - 6x_M + 2)))^2 + ((-(y_M(4x_N - 2) + (x_M - x_N)(y_C + 2y_D + 2y_E + y_F + 3^{1/2}(x_C - x_F)))/(2x_N - 6x_M + 2)) - (-(y_M(4x_N - 2) + (x_M - x_N)(y_E + 2y_F + 3^{1/2}(x_E - 1)))/(2x_N - 6x_M + 2)))^2) = 0$
which proves the claim as desired.

Case 2: the triangles are all internal to the hexagon.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates (x_E, y_E).
F has coordinates (x_F, y_F).
G has coordinates $(1/2, 3^{1/2}/2)$.
H has coordinates $(x_C/2 - (3^{1/2}y_C)/2 + 1/2, y_C/2 + (3^{1/2}(x_C - 1))/2)$.
I has coordinates
$(x_C/2 + x_D/2 + (3^{1/2}y_C)/2 - (3^{1/2}y_D)/2, y_C/2 + y_D/2 - (3^{1/2}(x_C - x_D))/2)$.
J has coordinates
$(x_E/2 + x_D/2 - (3^{1/2}y_E)/2 + (3^{1/2}y_D)/2, y_E/2 + y_D/2 + (3^{1/2}(x_E - x_D))/2)$.
K has coordinates
$(x_E/2 + x_F/2 + (3^{1/2}y_E)/2 - (3^{1/2}y_F)/2, y_E/2 + y_F/2 - (3^{1/2}(x_E - x_F))/2)$.
L has coordinates
$(x_F/2 + (3^{1/2}y_F)/2, y_F/2 - (3^{1/2}x_F)/2)$.
M has coordinates (x_M, y_M).
N has coordinates $(x_N, y_M + (x_N - x_M)(\frac{3^{1/2}/2 - y_M}{1/2 - x_M}))$.
O has coordinates $(-(x_N(x_C - 2x_M + 1) - x_C x_M + 3^{1/2}y_C(x_M - x_N))/(2x_M - 1), (y_M(2x_N - 1) + (y_C + 3^{1/2}(x_C - 1))(x_M - x_N))/(2x_M - 1))$.
P has coordinates
$((x_M(x_C + x_D - 1) - x_N(x_C + x_D - 2x_M) + 3^{1/2}y_C(x_M - x_N) - 3^{1/2}y_D(x_M - x_N))/(2x_M - 1), (y_M(2x_N - 1) + (x_M - x_N)(y_C + y_D - 3^{1/2}(x_C - x_D)))/(2x_M - 1))$.
Q has coordinates
$((1/2)(x_N + (-(x_N(x_C - 2x_M + 1) - x_C x_M + 3^{1/2}y_C(x_M - x_N))/(2x_M - 1))), (y_M(4x_N - 2) + (y_C + 3^{1/2}x_C)(x_M - x_N))/(4x_M - 2))$.
R has coordinates
$((1/2)(x_N + (x_M(x_C + x_D - 1) - x_N(x_C + x_D - 2x_M) + 3^{1/2}y_C(x_M - x_N) - 3^{1/2}y_D(x_M - x_N))/(2x_M - 1)), (y_M(4x_N - 2) + (x_M - x_N)(y_C + y_D + 3^{1/2}(x_D - x_C + 1)))/(4x_M - 2))$.
S has coordinates

$(-(x_N(2x_C + x_D - 4x_M + 1) - x_M(2x_C + x_D - 1) + 3^{1/2}y_D(x_M - x_N))/(4x_M - 2), (y_M(4x_N - 2) + (x_M - x_N)(2y_C + y_D + 3^{1/2}(x_D - 1)))/(4x_M - 2))$.
T has coordinates
$((x_N(2x_C + x_D - 4x_M + 2) - x_M(2x_C + x_D) + 3^{1/2}y_D(x_M - x_N))/(2x_N - 6x_M + 2), -(y_M(4x_N - 2) + (x_M - x_N)(2y_C + y_D + 3^{1/2}x_D))/(2x_N - 6x_M + 2))$.
U has coordinates
$((x_M(x_D + x_E - 1) - x_N(x_D + x_E - 2x_M) + 3^{1/2}y_D(x_M - x_N) - 3^{1/2}y_E(x_M - x_N))/(2x_M - 1), (y_M(2x_N - 1) + (x_M - x_N)(y_D + y_E - 3^{1/2}(x_D - x_E)))/(2x_M - 1))$.
V has coordinates
$((x_M(x_E + x_F - 1) - x_N(x_E + x_F - 2x_M) + 3^{1/2}y_E(x_M - x_N) - 3^{1/2}y_F(x_M - x_N))/(2x_M - 1), (y_M(2x_N - 1) + (x_M - x_N)(y_E + y_F - 3^{1/2}(x_E - x_F)))/(2x_M - 1))$.
W has coordinates
$(-(x_N(x_C + 2x_D + x_E - 4x_M) - x_M(x_C + 2x_D + x_E - 2) - 3^{1/2}y_C(x_M - x_N) + 3^{1/2}y_E(x_M - x_N))/(4x_M - 2), (y_M(4x_N - 2) + (x_M - x_N)(y_C + 2y_D + y_E - 3^{1/2}(x_C - x_E)))/(4x_M - 2))$.
Z has coordinates
$((x_M(x_C + x_D - 1) + x_M(x_E + x_F - 1) - x_N(x_C + x_D - 2x_M) - x_N(x_E + x_F - 2x_M) + 3^{1/2}y_C(x_M - x_N) - 3^{1/2}y_D(x_M - x_N) + 3^{1/2}y_E(x_M - x_N) - 3^{1/2}y_F(x_M - x_N))/(4x_M - 2), (y_M(4x_N - 2) + (x_M - x_N)(y_C + y_D + y_E + y_F - 3^{1/2}(x_C - x_D + x_E - x_F)))/(4x_M - 2))$.
A_1 has coordinates
$(-(x_N(x_D + 2x_E + x_F - 4x_M) - x_M(x_D + 2x_E + x_F - 2) - 3^{1/2}y_D(x_M - x_N) + 3^{1/2}y_F(x_M - x_N))/(4x_M - 2), (y_M(4x_N - 2) + (x_M - x_N)(y_D + 2y_E + y_F - 3^{1/2}(x_D - x_F)))/(4x_M - 2))$.
B_1 has coordinates
$((x_N(x_C + 2x_D + 2x_E + x_F - 4x_M) - x_M(x_C + 2x_D + 2x_E + x_F - 2) - 3^{1/2}y_C(x_M - x_N) + 3^{1/2}y_F(x_M - x_N))/(2x_N - 6x_M + 2), -(y_M(4x_N - 2) + (x_M - x_N)(y_C + 2y_D + 2y_E + y_F - 3^{1/2}(x_C - x_F)))/(2x_N - 6x_M + 2))$.
C_1 has coordinates
$((x_M(x_F - 1) - x_N(x_F - 2x_M) + 3^{1/2}y_F(x_M - x_N))/(2x_M - 1), (y_M(2x_N - 1) + (y_F - 3^{1/2}x_F)(x_M - x_N))/(2x_M - 1))$.
D_1 has coordinates
$((1/2)(x_N + (x_M(x_E + x_F - 1) - x_N(x_E + x_F - 2x_M) + 3^{1/2}y_E(x_M - x_N) - 3^{1/2}y_F(x_M - x_N))/(2x_M - 1)), (y_M(4x_N - 2) + (x_M - x_N)(y_E + y_F + 3^{1/2}(x_F - x_E + 1)))/(4x_M - 2))$.
E_1 has coordinates
$((1/2)(x_N + (x_M(x_F - 1) - x_N(x_F - 2x_M) + 3^{1/2}y_F(x_M - x_N))/(2x_M - 1)), (y_M(4x_N - 2) + (y_F - 3^{1/2}(x_F - 1))(x_M - x_N))/(4x_M - 2))$.
F_1 has coordinates
$((x_M(x_E + 2x_F - 2) - x_N(x_E + 2x_F - 4x_M) + 3^{1/2}y_E(x_M - x_N))/(4x_M - 2), (y_M(4x_N - 2) + (x_M - x_N)(y_E + 2y_F - 3^{1/2}x_E))/(4x_M - 2))$.
G_1 has coordinates
$((3^{1/2}(x_N - x_M)y_E - (4x_M - 2x_F - x_E - 1)x_N - (2x_F + x_E - 1)x_M)/(2(x_N - 3x_M + 1)), -(2(2x_N - 1)y_M - (x_N - x_M)(2y_F + y_E - 3^{1/2}(x_E - 1)))/(2(x_N - 3x_M + 1)))$.
We verify that segments TB_1 and TG_1 are equal in length since we calculate
$(x_T - x_{B_1})^2 + (y_T - y_{B_1})^2 - ((x_T - x_{G_1})^2 + (y_T - y_{G_1})^2)$
$= (((x_N(2x_C + x_D - 4x_M + 2) - x_M(2x_C + x_D) + 3^{1/2}y_D(x_M - x_N))/(2x_N - 6x_M + 2)) - ((x_N(x_C + 2x_D + 2x_E + x_F - 4x_M) - x_M(x_C + 2x_D + 2x_E + x_F - 2) - 3^{1/2}y_C(x_M - x_N) + 3^{1/2}y_F(x_M - x_N))/(2x_N - 6x_M + 2)))^2 + ((-(y_M(4x_N - 2) + (x_M - x_N)(2y_C + y_D + 3^{1/2}x_D))/(2x_N - 6x_M + 2)) - (-(y_M(4x_N - 2) + (x_M - x_N)(y_C + 2y_D + 2y_E + y_F - 3^{1/2}(x_C - x_F)))/(2x_N - 6x_M + 2)))^2 - ((((x_N(2x_C + x_D - 4x_M + 2) - x_M(2x_C +$

$x_D) + 3^{1/2}y_D(x_M - x_N))/(2x_N - 6x_M + 2)) - ((3^{1/2}(x_N - x_M)y_E - (4x_M - 2x_F - x_E - 1)x_N - (2x_F + x_E - 1)x_M)/(2(x_N - 3x_M + 1))))^2 + ((-(y_M(4x_N - 2) + (x_M - x_N)(2y_C + y_D + 3^{1/2}x_D))/(2x_N - 6x_M + 2)) - (-(2(2x_N - 1)y_M - (x_N - x_M)(2y_F + y_E - 3^{1/2}(x_E - 1)))/(2(x_N - 3x_M + 1))))^2) = 0$
and we verify that segments TB_1 and B_1G_1 are equal in length since we calculate
$(x_T - x_{B_1})^2 + (y_T - y_{B_1})^2 - ((x_{B_1} - x_{G_1})^2 + (y_{B_1} - y_{G_1})^2)$
$= (((x_N(2x_C + x_D - 4x_M + 2) - x_M(2x_C + x_D) + 3^{1/2}y_D(x_M - x_N))/(2x_N - 6x_M + 2)) - ((x_N(x_C + 2x_D + 2x_E + x_F - 4x_M) - x_M(x_C + 2x_D + 2x_E + x_F - 2) - 3^{1/2}y_C(x_M - x_N) + 3^{1/2}y_F(x_M - x_N))/(2x_N - 6x_M + 2)))^2 + ((-(y_M(4x_N - 2) + (x_M - x_N)(2y_C + y_D + 3^{1/2}x_D))/(2x_N - 6x_M + 2)) - (-(y_M(4x_N - 2) + (x_M - x_N)(y_C + 2y_D + 2y_E + y_F - 3^{1/2}(x_C - x_F)))/(2x_N - 6x_M + 2)))^2 - ((((x_N(x_C + 2x_D + 2x_E + x_F - 4x_M) - x_M(x_C + 2x_D + 2x_E + x_F - 2) - 3^{1/2}y_C(x_M - x_N) + 3^{1/2}y_F(x_M - x_N))/(2x_N - 6x_M + 2)) - ((3^{1/2}(x_N - x_M)y_E - (4x_M - 2x_F - x_E - 1)x_N - (2x_F + x_E - 1)x_M)/(2(x_N - 3x_M + 1))))^2 + ((-(y_M(4x_N - 2) + (x_M - x_N)(y_C + 2y_D + 2y_E + y_F - 3^{1/2}(x_C - x_F)))/(2x_N - 6x_M + 2)) - (-(2(2x_N - 1)y_M - (x_N - x_M)(2y_F + y_E - 3^{1/2}(x_E - 1)))/(2(x_N - 3x_M + 1))))^2) = 0$
which proves the claim as desired.

□

The Jha-Savaran theorem follows as the special case of this by taking $N = G$.

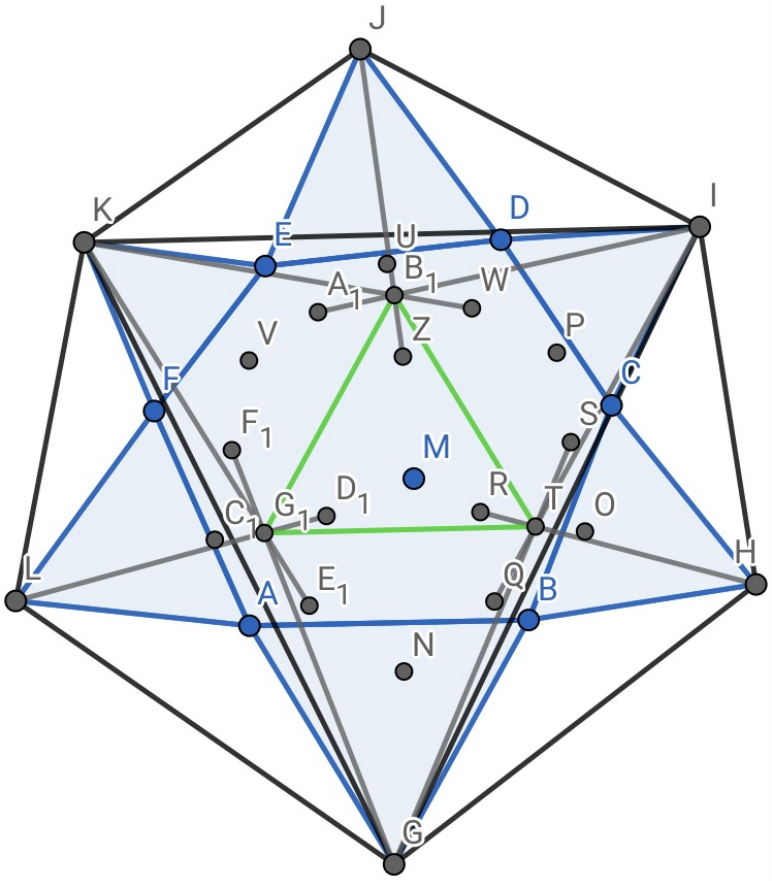

FIGURE 1. Illustration of theorem 2.1. where we take N to be the midpoint of G and M.

References

[1] Villiers, M. D., Humenberger, H., Schuppar, B., "Jha and Savaran's Generalisation of Napoleon's Theorem", *Global Journal of Advanced Research on Classical and Modern Geometries*, ISSN: 2284-5569, Vol.11, (2022), Issue 2, pp.190-197

[2] Weisstein, Eric W. "Napoleon's Theorem." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/NapoleonsTheorem.html

[3] Warren, B.L., "Generalization of the Triangle Centroid and the Van Lamoen Circle", *International Journal of Computer Discovered Mathematics*, Vol. 9, (2024), pp. 45-54.