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A Seven Circumcenters Theorem

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1. Introduction

In a given triangle, pick a point and construct the cevians. We prove that seven particular circumcenters associated to this construction give rise to three particular lines which are either all parallel or concurrent.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be some point in the plane.
Let lines AB and CD intersect at point E.
Let lines AC and BD intersect at point F.
Let lines AD and BC intersect at point G.
Let H be the circumcenter of A, E, and F.
Let I be the circumcenter of B, E, and G.
Let J be the circumcenter of C, F, and G.
Let K be the circumcenter of E, F, and G.
Let L be the circumcenter of H, I, and K.
Let M be the circumcenter of H, J, and K.
Let N be the circumcenter of I, J, and K.
Then lines EL, FM, and GN are either parallel or concurrent.

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Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates $((x_C * y_D - x_D * y_C)/(y_D - y_C), 0)$.
F has coordinates $((x_C * y_D)/(x_C * y_D - x_D * y_C + y_C), (y_C * y_D)/(x_C * y_D - x_D * y_C + y_C))$.
G has coordinates $((x_D * y_C)/(-x_C * y_D + y_D + x_D * y_C), (y_C * y_D)/(-x_C * y_D + y_D + x_D * y_C))$.
H has coordinates
$((x_C*y_D-x_D*y_C)/(2*(y_D-y_C)), (x_C*((x_C*y_D)/(2*x_C*y_D-2*x_D*y_C+2*y_C)+(x_C*y_D-x_D*y_C)/(2*y_C-2*y_D)))/y_C+(y_C*y_D)/(2*x_C*y_D-2*x_D*y_C+2*y_C))$.
I has coordinates
$(((x_C*y_D-x_D*y_C)/(y_D-y_C)+1)/2, -((y_C^2+(x_C-1)^2*x_C)*y_D^2-y_C*(y_C^2+2*(x_C-1)^2*x_D)*y_D+(x_C-1)*(x_D-1)*x_D*y_C^2)/(2*y_C*(y_D-y_C)*((x_C-1)*y_D-x_D*y_C)))$.
J has coordinates
$((((2*x_C-1)*y_C^2+2*x_C^3-2*x_C^2)*y_D^2-y_C*((2*x_D-1)*y_C^2+x_C*(x_C*(2*x_D-1)+1))*y_D+(x_D-1)*x_D*y_C^2)/(2*((x_C-1)*y_D-x_D*y_C)*(x_C*y_D-(x_D-1)*y_C)), -(((x_C^2-x_C+1)*y_C^2+x_C^4-2*x_C^3+x_C^2)*y_D^2+y_C^3*y_D+(1-x_D)*x_D*y_C^4+(1-x_C)*x_C*(x_D-1)*x_D*y_C^2)/(2*y_C*((x_C-1)*y_D-x_D*y_C)*(x_C*y_D-(x_D-1)*y_C)))$.
K has coordinates
$(-(((2*x_C-1)*y_C^2-2*x_C^5+3*x_C^4-x_C^2)*y_D^5-y_C*((2*x_D+4*x_C-3)*y_C^2-(x_C-1)*x_C^2*(2*(5*x_C-1)*x_D-3*x_C-1))*y_D^4+y_C^2*((4*x_D+2*x_C-3)*y_C^2-2*x_C*x_D*((10*x_C^2-9*x_C+1)*x_D-(2*x_C-1)*(3*x_C-1)))*y_D^3-y_C^3*((2*x_D-1)*y_C^2-2*x_C*x_D*(x_D*(2*(5*x_C-3)*x_D-9*x_C+5)+x_C-1))*y_D^2+(1-x_D)*x_D^2*((10*x_C-3)*x_D-2*x_C-1)*y_C^4*y_D+(x_D-1)^2*x_D^2*(2*x_D+1)*y_C^5)/(4*(y_D-y_C)*((x_C-1)*y_D-(x_D-1)*y_C)*((x_C-1)*y_D-x_D*y_C)*(x_C*y_D-(x_D-1)*y_C)*(x_C*y_D-x_D*y_C)), -((x_C-1)*x_C*(y_C^2+(x_C-1)^2*x_C^2)*y_D^6-2*y_C*(2*x_C*x_D*y_C^2-x_D*y_C^2-x_C^2*y_C^2+2*x_C^5*x_D-5*x_C^4*x_D+4*x_C^3*x_D-x_C^2*x_D-x_C^5+2*x_C^4-x_C^3)*y_D^5+y_C^2*(3*x_D^2*y_C^2-3*x_D*y_C^2-3*x_C^2*y_C^2+3*x_C*y_C^2+5*x_C^4*x_D^2-10*x_C^3*x_D^2+5*x_C^2*x_D^2-5*x_C^4*x_D+10*x_C^3*x_D-5*x_C^2*x_D)*y_D^4-2*(x_D^2-2*x_C*x_D+x_C)*y_C^5*y_D^3+(1-x_D)*x_D*y_C^4*(y_C^2+5*(x_C-1)*x_C*(x_D-1)*x_D)*y_D^2+2*(x_D-1)^2*x_D^2*((2*x_C-1)*x_D-x_C)*y_C^5*y_D-(x_D-1)^3*x_D^3*y_C^6)/(4*y_C*y_D*(y_D-y_C)*((x_C-1)*y_D-(x_D-1)*y_C)*((x_C-1)*y_D-x_D*y_C)*(x_C*y_D-(x_D-1)*y_C)*(x_C*y_D-x_D*y_C)))$.
L has coordinates
$x_L = -((y_I - y_H) * y_K^2 + (-y_I^2 + y_H^2 - x_I^2 + x_H^2) * y_K + y_H * y_I^2 + (-y_H^2 + x_K^2 - x_H^2) * y_I + (x_I^2 - x_K^2) * y_H)/(2 * ((x_I - x_H) * y_K + (x_H - x_K) * y_I + (x_K - x_I) * y_H))$
and
$y_L = ((x_I - x_H) * y_K^2 + (x_H - x_K) * y_I^2 + (x_K - x_I) * y_H^2 + (x_I - x_H) * x_K^2 + (x_H^2 - x_I^2) * x_K + x_H * x_I^2 - x_H^2 * x_I)/(2*((x_I - x_H) * y_K + (x_H - x_K) * y_I + (x_K - x_I) * y_H))$.
M has coordinates
$x_M = -((y_J - y_H) * y_K^2 + (-y_J^2 + y_H^2 - x_J^2 + x_H^2) * y_K + y_H * y_J^2 + (-y_H^2 + x_K^2 - x_H^2) * y_J + (x_J^2 - x_K^2) * y_H)/(2*((x_J - x_H) * y_K + (x_H - x_K) * y_J + (x_K - x_J) * y_H))$
and
$y_M = ((x_J - x_H) * y_K^2 + (x_H - x_K) * y_J^2 + (x_K - x_J) * y_H^2 + (x_J - x_H) * x_K^2 + (x_H^2 - x_J^2) * x_K + x_H * x_J^2 - x_H^2 * x_J)/(2*((x_J - x_H) * y_K + (x_H - x_K) * y_J + (x_K - x_J) * y_H))$.
N has coordinates
$x_N = -((y_J - y_I) * y_K^2 + (-y_J^2 + y_I^2 - x_J^2 + x_I^2) * y_K + y_I * y_J^2 + (-y_I^2 + x_K^2 - x_I^2) *$

$y_J + (x_J^2 - x_K^2) * y_I)/(2 * ((x_J - x_I) * y_K + (x_I - x_K) * y_J + (x_K - x_J) * y_I))$
and
$y_N = ((x_J - x_I) * y_K^2 + (x_I - x_K) * y_J^2 + (x_K - x_J) * y_I^2 + (x_J - x_I) * x_K^2 + (x_I^2 - x_J^2) * x_K + x_I * x_J^2 - x_I^2 * x_J)/(2 * ((x_J - x_I) * y_K + (x_I - x_K) * y_J + (x_K - x_J) * y_I))$.
Line EL is given by the equation $x(\frac{y_L - y_E}{x_L - x_E}) - y + y_E - x_E(\frac{y_L - y_E}{x_L - x_E}) = 0$.
Line FM is given by the equation $x(\frac{y_M - y_F}{x_M - x_F}) - y + y_F - x_F(\frac{y_M - y_F}{x_M - x_F}) = 0$.
Line GN is given by the equation $x(\frac{y_N - y_G}{x_N - x_G}) - y + y_G - x_G(\frac{y_N - y_G}{x_N - x_G}) = 0$.
Suppose that D is chosen so that these three lines are not parallel.
One can then substitute the above values into the following matrix and calculate that

$$det \begin{bmatrix} \frac{y_L - y_E}{x_L - x_E} & -1 & y_E - x_E(\frac{y_L - y_E}{x_L - x_E}) \\ \frac{y_M - y_F}{x_M - x_F} & -1 & y_F - x_F(\frac{y_M - y_F}{x_M - x_F}) \\ \frac{y_N - y_G}{x_N - x_G} & -1 & y_G - x_G(\frac{y_N - y_G}{x_N - x_G}) \end{bmatrix} = 0$$

which verifies that lines EL, FM, and GN are indeed concurrent as desired. □

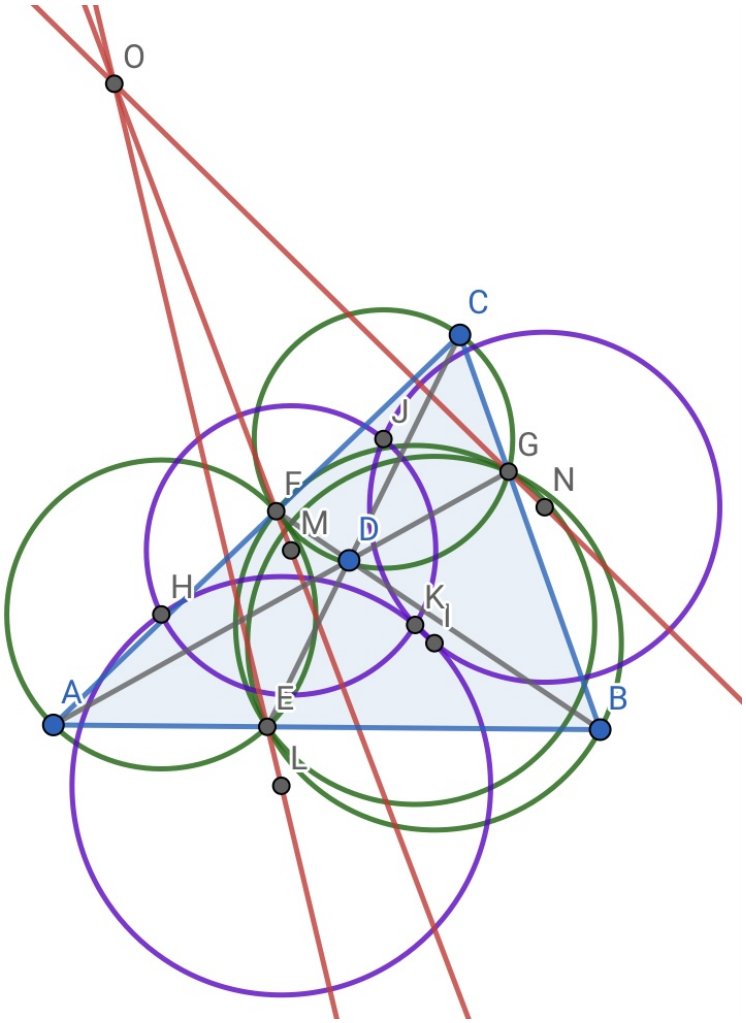

Figure 1. Illustration.

References

[1] Warren, B.L., "An Extension of the Van Lamoen Circle", *International Journal of Computer Discovered Mathematics*, Vol. 9, (2024), pp. 79-82.