International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 9, 2024, pp. 230-233.
Received July 11, 2024. Published on-line December 31, 2024.
web: http://www.journal-1.eu/
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The Seven Circumcenters Theorem

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Mathematics Subject Classification (2010). 51N20.

1. Introduction

Miquel's theorem [1] is an example of a theorem where the construction involves choosing a random point on each side of a given triangle. Here, we will build the same construction as in Miquel's theorem but will demonstrate the existence of a different point than the Miquel point.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be a point on line AB.
Let E be a point on line AC.
Let F be a point on line BC.
Let G be the circumcenter of A, D, and E.
Let H be the circumcenter of B, D, and F.
Let I be the circumcenter of C, E, and F.
Let J be the circumcenter of D, E, and F.
Let K be the circumcenter of G, I, and J.
Let L be the circumcenter of H, I, and J.
Let M be the circumcenter of G, H, and J.
Then lines DM, EK, and FL are concurrent.

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Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates $(x_D, 0)$.
E has coordinates $(x_E, \frac{x_E y_C}{x_C})$.
F has coordinates $(x_F, (x_F - 1)(\frac{y_C}{x_C - 1}))$.
G has coordinates
$(x_D/2, (x_E * y_C)/(2 * x_C) - (x_C * (x_D/2 - x_E/2))/y_C)$.
H has coordinates
$((x_D + 1)/2, ((x_F - 1) * y_C^2 + (x_C - 1)^2 * x_F - (x_C - 1)^2 * x_D)/(2 * (x_C - 1) * y_C))$.
I has coordinates
$(-((x_C * x_F - (x_C - 1) * x_E - x_C) * y_C^2 + (x_C - 1)^2 * x_C * x_F - (x_C - 1) * x_C^2 * x_E - x_C^3 + x_C^2)/(2 * (x_C - 1) * x_C), ((x_C^2 * x_F - (x_C - 1)^2 * x_E - x_C) * y_C)/(2 * (x_C - 1) * x_C) + ((x_C - 1) * x_C * x_F - (x_C - 1) * x_C * x_E)/(2 * y_C))$.
J has coordinates
$(-(x_E * (x_F - 1) * (x_C * x_F - x_C * x_E + x_E - x_C) * y_C^2 + (x_C - 1)^2 * x_C * x_E * x_F^2 - (x_C - 1) * x_C^2 * (x_E - x_D) * (x_E + x_D) * x_F + (x_C - 1) * x_C^2 * x_E^2 - (x_C - 1)^2 * x_C * x_D^2 * x_E - (x_C - 1) * x_C^2 * x_D^2)/(2 * (x_C - 1) * x_C * ((x_E - x_C * x_D) * x_F + ((x_C - 1) * x_D - x_C) * x_E + x_C * x_D)), ((x_C^2 * (x_E - x_D) * x_F^2 - (x_C^2 * x_E^2 - 2 * x_C * x_E^2 + x_E^2 + 2 * x_C^2 * x_E - 2 * x_C^2 * x_D) * x_F + (x_C - 1)^2 * x_D * x_E^2 + x_C^2 * x_E - x_C^2 * x_D) * y_C^2 + (x_C - 1)^2 * x_C^2 * (x_E - x_D) * x_F^2 - (x_C - 1)^2 * x_C^2 * (x_E - x_D) * (x_E + x_D) * x_F + (x_C - 1)^2 * x_C^2 * x_D * x_E^2 - (x_C - 1)^2 * x_C^2 * x_D^2 * x_E)/(2 * (x_C - 1) * x_C * ((x_E - x_C * x_D) * x_F + ((x_C - 1) * x_D - x_C) * x_E + x_C * x_D) * y_C))$.
K has coordinates
$x_K = -((x_C * x_F - x_C * x_E + x_E - x_C) * (x_C * x_E * x_F + x_E * x_F - x_C * x_D * x_F - x_C * x_E^2 + x_E^2 + x_C * x_D * x_E - x_D * x_E - x_C * x_E - x_E + x_C * x_D) * y_C^4 + (x_C - 1) * x_C * (2 * x_C^2 * x_E * x_F^2 - x_E * x_F^2 - 2 * x_C^2 * x_D * x_F^2 + x_C * x_D * x_F^2 - 4 * x_C^2 * x_E^2 * x_F + 2 * x_C * x_E^2 * x_F - x_E^2 * x_F + 4 * x_C^2 * x_D * x_E * x_F - 2 * x_C * x_D * x_E * x_F - 2 * x_C^2 * x_E * x_F + x_C * x_D^2 * x_F + 2 * x_C^2 * x_D * x_F + 2 * x_C^2 * x_E^3 - 2 * x_C * x_E^3 - 2 * x_C^2 * x_D * x_E^2 + x_C * x_D * x_E^2 + x_D * x_E^2 + 2 * x_C^2 * x_E^2 + x_C * x_E^2 - x_C * x_D^2 * x_E + x_D^2 * x_E - 2 * x_C^2 * x_D * x_E + x_C^2 * x_E - x_C * x_D^2 - x_C^2 * x_D) * y_C^2 + (x_C - 1)^3 * x_C^3 * (x_E - x_D) * x_F^2 - (x_C - 1)^2 * x_C^3 * (x_E - x_D) * ((2 * x_C - 1) * x_E - x_D) * x_F + (x_C - 1)^2 * x_C^4 * x_E^3 - (x_C - 1)^2 * x_C^3 * (x_C + 1) * x_D * x_E^2 + (x_C - 1)^2 * x_C^3 * x_D^2 * x_E)/(4 * (x_C - 1) * x_C * (x_E * x_F - x_C * x_D * x_F + x_C * x_D * x_E - x_D * x_E - x_C * x_E + x_C * x_D) * y_C^2)$
and
$y_K = -(x_E * (x_C * x_F - x_C * x_E + x_E - x_C)^2 * y_C^4 + x_C * (2 * x_C^3 * x_E * x_F^2 - 4 * x_C^2 * x_E * x_F^2 + x_C * x_E * x_F^2 + x_C^2 * x_D * x_F^2 - 4 * x_C^3 * x_E^2 * x_F + 8 * x_C^2 * x_E^2 * x_F - 7 * x_C * x_E^2 * x_F + 3 * x_E^2 * x_F + 2 * x_C^2 * x_D * x_E * x_F - 2 * x_C * x_D * x_E * x_F - 2 * x_C^3 * x_E * x_F + 4 * x_C^2 * x_E * x_F - 2 * x_C^2 * x_D * x_F + 2 * x_C^3 * x_E^3 - 4 * x_C^2 * x_E^3 + 2 * x_C * x_E^3 - 3 * x_C^2 * x_D * x_E^2 + 6 * x_C * x_D * x_E^2 - 3 * x_D * x_E^2 + 2 * x_C^3 * x_E^2 - x_C^2 * x_E^2 - x_C * x_E^2 - 2 * x_C^2 * x_D * x_E + 2 * x_C * x_D * x_E + x_C^3 * x_E - 2 * x_C^2 * x_E + x_C^2 * x_D) * y_C^2 + (x_C - 1)^2 * x_C^3 * ((x_C - 2) * x_E + x_D) * x_F^2 - (x_C - 1) * x_C^3 * (2 * x_C^2 * x_E^2 - 4 * x_C * x_E^2 + 3 * x_E^2 - 2 * x_C * x_D * x_E + 2 * x_C * x_D^2 - x_D^2) * x_F + (x_C - 1)^2 * x_C^4 * x_E^3 - (x_C - 1) * x_C^3 * (3 * (x_C - 1) * x_D - x_C) * x_E^2 + 2 * (x_C - 1) * x_C^3 * x_D * ((x_C - 1) * x_D - x_C) * x_E + (x_C - 1) * x_C^4 * x_D^2)/(4 * (x_C - 1) * x_C^2 * (x_E * x_F - x_C * x_D * x_F + x_C * x_D * x_E - x_D * x_E - x_C * x_E + x_C * x_D) * y_C)$.
L has coordinates
$x_L = -((x_C * x_F - x_C * x_E + x_E - x_C) * (x_C * x_F^2 - x_C * x_E * x_F + 2 * x_E * x_F - x_C * x_D * x_F - x_C * x_F + x_C * x_D * x_E - x_D * x_E - x_E + x_C * x_D) * y_C^4 + (x_C - 1) * x_C * (2 * x_C^2 * x_F^3 - 2 * x_C * x_F^3 - 4 * x_C^2 * x_E * x_F^2 + 6 * x_C * x_E * x_F^2 - 3 * x_E * x_F^2 - 2 * x_C^2 * x_D * x_F^2 + 3 * x_C * x_D * x_F^2 - 2 * x_C^2 * x_F^2 + 2 * x_C * x_F^2 + 2 * x_C^2 * x_E^2 * x_F - 4 * x_C * x_E^2 * x_F + x_E^2 * x_F + 4 * x_C^2 * x_D * x_E * x_F - 6 * x_C * x_D * x_E * x_F + 2 * x_D * x_E * x_F + 2 * x_C^2 * x_E * x_F - 2 * x_C * x_E *$

$x_F+x_C*x_D^2*x_F+2*x_C^2*x_D*x_F-2*x_C*x_D*x_F+x_C^2*x_F-x_C*x_F-2*x_C^2*x_D*x_E^2+3*x_C*x_D*x_E^2-x_D*x_E^2+x_C*x_E^2-x_C*x_D^2*x_E+x_D^2*x_E-2*x_C^2*x_D*x_E+2*x_C*x_D*x_E+x_C*x_E-x_C*x_D^2-x_C^2*x_D)*y_C^2+(x_C-1)^4*x_C^2*x_F^3-(x_C-1)^3*x_C^2*((2*x_C-1)*x_E+(x_C-2)*x_D)*x_F^2+(x_C-1)^3*x_C^2*(x_C*x_E^2+2*x_C*x_D*x_E-2*x_D*x_E-x_D^2)*x_F-(x_C-1)^3*x_C^3*x_D*x_E^2+(x_C-1)^3*x_C^2*x_D^2*x_E)/(4*(x_C-1)*x_C*(x_E*x_F-x_C*x_D*x_F+x_C*x_D*x_E-x_D*x_E-x_C*x_E+x_C*x_D)*y_C^2)$
and
$y_L=-((x_F-1)*(x_C*x_F-x_C*x_E+x_E-x_C)^2*y_C^4+(x_C-1)*(2*x_C^3*x_F^3-2*x_C^2*x_F^3-4*x_C^3*x_E*x_F^2+4*x_C^2*x_E*x_F^2-3*x_C*x_E*x_F^2+3*x_C^2*x_D*x_F^2-4*x_C^3*x_F^2+4*x_C^2*x_F^2+2*x_C^3*x_E^2*x_F-2*x_C^2*x_E^2*x_F-x_C*x_E^2*x_F+x_E^2*x_F-2*x_C^2*x_D*x_E*x_F+2*x_C*x_D*x_E*x_F+6*x_C^3*x_E*x_F-4*x_C^2*x_E*x_F+4*x_C*x_E*x_F-6*x_C^2*x_D*x_F+3*x_C^3*x_F-3*x_C^2*x_F-x_C^2*x_D*x_E^2+2*x_C*x_D*x_E^2-x_D*x_E^2-2*x_C^3*x_E^2+3*x_C^2*x_E^2-x_C*x_E^2+2*x_C^2*x_D*x_E-2*x_C*x_D*x_E-2*x_C^3*x_E-x_C*x_E+3*x_C^2*x_D-x_C^3+x_C^2)*y_C^2+(x_C-1)^4*x_C^2*x_F^3-(x_C-1)^3*x_C*((2*x_C^2+1)*x_E-x_C*(3*x_D-x_C+1))*x_F^2+(x_C-1)^3*x_C*(x_C^2*x_E^2+x_C*x_E^2-2*x_C*x_D*x_E+2*x_D*x_E+2*x_C^2*x_E-2*x_C*x_D^2-2*x_C*x_D)*x_F-(x_C-1)^3*x_C^2*(x_D+x_C)*x_E^2+(x_C-1)^3*x_C*(2*x_C-1)*x_D^2*x_E+(x_C-1)^3*x_C^2*x_D^2)/(4*(x_C-1)^2*x_C*(x_E*x_F-x_C*x_D*x_F+x_C*x_D*x_E-x_D*x_E-x_C*x_E+x_C*x_D)*y_C)$.
M has coordinates
$x_M=-(x_E*(x_F-1)*(x_C*x_F-x_C*x_E+x_E-x_C)*y_C^4+(x_C-1)*x_C*(2*x_C*x_E*x_F^2-x_E*x_F^2-x_C*x_D*x_F^2-2*x_C*x_E^2*x_F+x_E^2*x_F-2*x_D*x_E*x_F-2*x_C*x_E*x_F+3*x_C*x_D^2*x_F+2*x_C*x_D*x_F+x_C*x_D*x_E^2-x_D*x_E^2+x_C*x_E^2-3*x_C*x_D^2*x_E+3*x_D^2*x_E+2*x_C*x_D*x_E+x_C*x_E-3*x_C*x_D^2-x_C*x_D)*y_C^2+(x_C-1)^3*x_C^2*(x_E-x_D)*x_F^2-(x_C-1)^2*x_C^2*(x_E-x_D)*(x_C*x_E+x_C*x_D-2*x_D)*x_F+(x_C-1)^2*x_C^3*x_D*x_E^2-(x_C-1)^2*x_C^2*(x_C+1)*x_D^2*x_E+(x_C-1)^2*x_C^2*x_D^3)/(4*(x_C-1)*x_C*(x_E*x_F-x_C*x_D*x_F+x_C*x_D*x_E-x_D*x_E-x_C*x_E+x_C*x_D)*y_C^2)$
and
$y_M=((x_C*(x_E-x_C*x_D)*x_F^2+((x_C-1)*x_E^2-2*x_C*x_E+2*x_C^2*x_D)*x_F+(x_C-1)*((x_C-1)*x_D-x_C)*x_E^2+x_C*x_E-x_C^2*x_D)*y_C^2+(x_C-1)^2*x_C*(x_E-x_C*x_D)*x_F^2+(x_C-1)*x_C*(x_C*x_E^2-4*x_C*x_D*x_E+2*x_D*x_E+2*x_C^2*x_D^2-x_C*x_D^2)*x_F+(x_C-1)*x_C^2*((x_C-1)*x_D-x_C)*x_E^2-(x_C-1)*x_C*x_D*((x_C-1)*(2*x_C-1)*x_D-2*x_C^2)*x_E-(x_C-1)*x_C^3*x_D^2)/(4*(x_C-1)*x_C*(x_E*x_F-x_C*x_D*x_F+x_C*x_D*x_E-x_D*x_E-x_C*x_E+x_C*x_D)*y_C)$.
Line DM is given by the equation
$x(\frac{y_M-y_D}{x_M-x_D})-y+y_D-x_D(\frac{y_M-y_D}{x_M-x_D})=0$.
Line EK is given by the equation
$x(\frac{y_K-y_E}{x_K-x_E})-y+y_E-x_E(\frac{y_K-y_E}{x_K-x_E})=0$.
Line FL is given by the equation
$x(\frac{y_L-y_F}{x_L-x_F})-y+y_F-x_F(\frac{y_L-y_F}{x_L-x_F})=0$.
Suppose that not all three lines are parallel.

One can then substitute the above values into the following matrix and calculate that

$$det\begin{bmatrix}\frac{y_M-y_D}{x_M-x_D} & -1 & y_D-x_D(\frac{y_M-y_D}{x_M-x_D})\\ \frac{y_K-y_E}{x_K-x_E} & -1 & y_E-x_E(\frac{y_K-y_E}{x_K-x_E})\\ \frac{y_L-y_F}{x_L-x_F} & -1 & y_F-x_F(\frac{y_L-y_F}{x_L-x_F})\end{bmatrix}=0$$

as desired. □

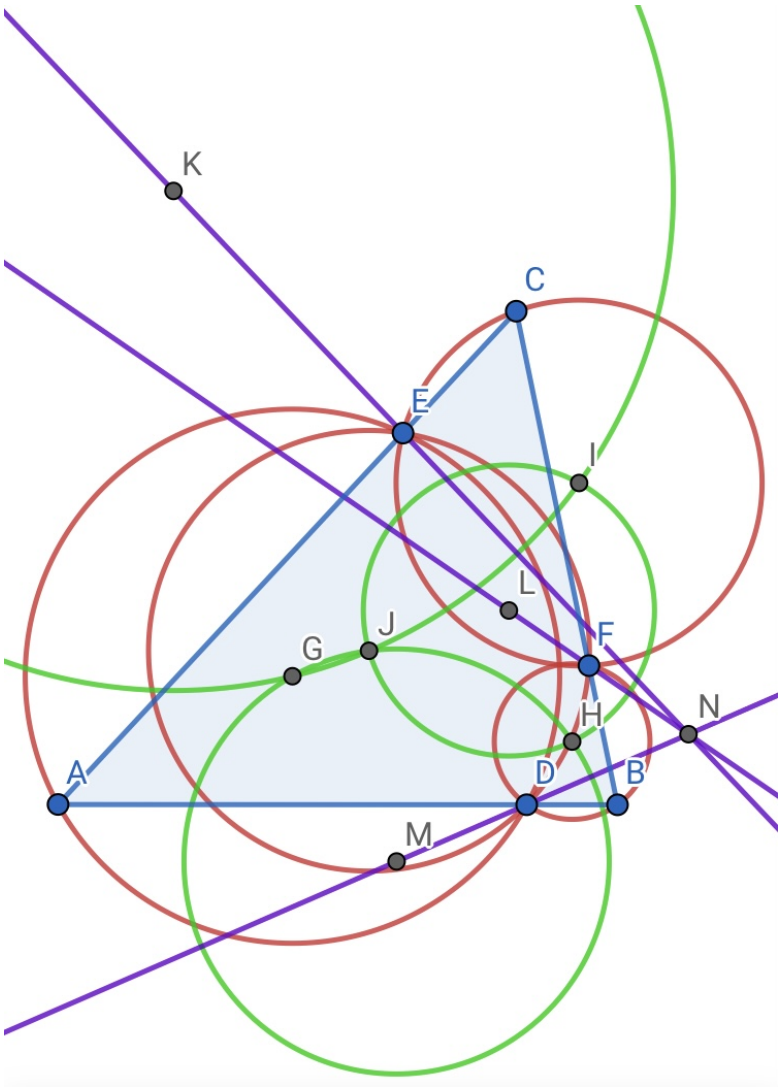

FIGURE 1. Illustration.

REFERENCES

[1] van Lamoen, Floor and Weisstein, Eric W. "Miquel Point." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/MiquelPoint.html