International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 9, 2024, pp. 374-380.
Received August 28, 2024. Published on-line December 31, 2024.
web: http://www.journal-1.eu/
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A Fourth Point on a Certain Generalized Euler Line

Benjamin Lee Warren[a] [2]
University of Binghamton, Mathematics,
4400 Vestel Pkwy E, Binghamton, NY 13902
e-mail: warrenbenjamin52@yahoo.com
web: https://orcid.org/0000-0002-2829-6953

Mathematics Subject Classification (2010). 51N20.

1. Introduction

We prove that the tangential triangle of the circumcevian triangle is in perspective with the original triangle and that the perspector lies on the line referred to in [1].

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be a point in the plane.
Let α be the tangent line (to the circumcircle of ABC) through A.
Let β be the tangent line through B.
Let γ be the tangent line through C.
Let α and β intersect at E.
Let α and γ intersect at F.
Let β and γ intersect at G.
Let line CD intersect the circumcircle of ABC at point H.
Let lines AB and CD intersect at I.
Let line BD intersect the circumcircle of ABC at point J.
Let lines AC and BD intersect at K.
Let lines AD and BC intersect at point L.
Let line AD intersect the circumcircle of ABC at M.
Let lines EH, FJ, and GM intersect at N.

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[2]Corresponding author

Let lines EI, FK, and GL intersect at$O$.
Let δ be the tangent line through H.
Let ε be the tangent line through J.
Let ζ be the tangent line through M.
Let δ and ε intersect at P.
Let ε and ζ intersect at Q.
Let δ and ζ intersect at R.
Then,
1. Lines AP, BR, and CQ intersect at S, and
2. S lies on line DNO.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0,0)$.
B has coordinates $(1,0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates $(\frac{1}{2}, -\frac{y_C}{2x_C^2-2x_C+2y_C^2})$.
F has coordinates $(\frac{x_C^2-x_C+y_C^2}{2(x_C-1)}, -\frac{y_C}{2(x_C-1)})$.
G has coordinates $(\frac{x_C^2+x_C+y_C^2}{2x_C}, \frac{y_C}{2x_C})$.
H has coordinates $(((x_Cy_D - x_Dy_C)(y_C(y_D-y_C)+(x_C-1)(x_D-x_C)))/(y_C((y_D-y_C)^2+(x_D-x_C)^2)), (((x_C-1)y_D-(x_D-1)y_C)(x_Cy_D-x_Dy_C))/(y_C((y_D-y_C)^2+(x_D-x_C)^2)))$.
I has coordinates $(\frac{x_Cy_D-x_Dy_C}{y_D-y_C}, 0)$.
J has coordinates $((y_D(y_C(y_D+(x_D-1)y_C)+(x_C-1)x_C(x_D-1)))/(y_C(y_D^2+(x_D-1)^2)), (y_D((y_C^2+(x_C-1)x_C)y_D-(x_D-1)y_C))/(y_C(y_D^2+(x_D-1)^2)))$.
K has coordinates $(\frac{x_Cy_D}{x_Cy_D-x_Dy_C+y_C}, \frac{y_Cy_D}{x_Cy_D-x_Dy_C+y_C})$.
L has coordinates $(\frac{x_Dy_C}{-x_Cy_D+x_Dy_C+y_D}, \frac{y_Cy_D}{-x_Cy_D+x_Dy_C+y_D})$.
M has coordinates $((x_D((y_C^2+(x_C-1)x_C)y_D+x_Dy_C))/(y_C(y_D^2+x_D^2)), (y_D((y_C^2+(x_C-1)x_C)y_D+x_Dy_C))/(y_C(y_D^2+x_D^2)))$.
N has coordintes
$((y_C((y_C^2+(x_C-1)^2)(y_C^2+x_C(x_C+1))(y_C^2+x_C^2)y_D^2(y_D^2+x_D^2)-(y_C^2+(x_C-1)x_C)(y_C^2+x_C(x_C+1))y_D((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_D^2+(x_D-1)^2)+x_Dy_C(y_C^2+(x_C-1)x_C)((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_D^2+(x_D-1)^2))+2(x_C-1)(y_C^2+x_C(x_C+1))y_D^2((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_C(y_D+(x_D-1)y_C)+(x_C-1)x_C(x_D-1))-2x_Cx_Dy_Cy_D((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_C(y_D+(x_D-1)y_C)+(x_C-1)x_C(x_D-1))-2(x_C-1)x_Dy_Cy_D((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_C(y_D+(x_D-1)y_C)+(x_C-1)x_C(x_D-1))-2x_Cx_D(y_C^2+(x_C-1)x_C)y_D((y_C^2+(x_C-1)x_C)y_D-(x_D-1)y_C)((y_C^2+(x_C-1)x_C)y_D+x_Dy_C))/(y_C((y_D^2+x_D^2)(y_C(y_C(y_C^2+x_C^2+x_C)+y_C(y_C^2+x_C^2-x_C))(y_D^2+(x_D-1)^2)-2(x_C-1)y_Cy_D(y_C(y_D+(x_D-1)y_C)+(x_C-1)x_C(x_D-1))+2(x_C-1)(y_C^2+x_C^2+x_C)y_D((y_C^2+(x_C-1)x_C)y_D-(x_D-1)y_C))-2x_C(y_C^2+x_C^2-x_C)y_D((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_D^2+(x_D-1)^2)-2x_Cx_Dy_C((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_D^2+(x_D-1)^2))+4(x_C-1)x_Cy_D^2((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_C(y_D+(x_D-1)y_C)+(x_C-1)x_C(x_D-1))-4(x_C-1)x_Cx_Dy_D((y_C^2+(x_C-1)x_C)y_D-(x_D-1)y_C)((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)), (y_C(y_C((y_C^2+(x_C-1)^2)(y_C^2+x_C^2)y_D^2(y_D^2+x_D^2)+(y_C^2+x_C(x_C+1))y_D((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_D^2+(x_D-1)^2)-x_Dy_C((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_D^2+(x_D-1)^2))-2(x_C-1)x_Dy_D((y_C^2+(x_C-1)x_C)y_D-(x_D-1)y_C)((y_C^2+(x_C-1)x_C)y_D+x_Dy_C))-2x_Cy_Cy_D^2((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_C(y_D+(x_D-1)y_C)+(x_C-1)x_C(x_D-1))+2(x_C-1)(y_C^2+x_C(x_C+1))y_D^2((y_C^2+(x_C-1)x_C)y_D-$

$(x_D-1)y_C)((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)-2x_C(y_C^2+(x_C-1)x_C)y_D^2((y_C^2+(x_C-1)x_C)y_D-(x_D-1)y_C)((y_C^2+(x_C-1)x_C)y_D+x_Dy_C))/(y_C((y_D^2+x_D^2)(y_C(y_C(y_C^2+x_C^2+x_C)+y_C(y_C^2+x_C^2-x_C))(y_D^2+(x_D-1)^2)-2(x_C-1)y_Cy_D(y_C(y_D+(x_D-1)y_C)+(x_C-1)x_C(x_D-1))+2(x_C-1)(y_C^2+x_C^2+x_C)y_D((y_C^2+(x_C-1)x_C)y_D-(x_D-1)y_C))-2x_C(y_C^2+x_C^2-x_C)y_D((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_D^2+(x_D-1)^2)-2x_Cx_Dy_C((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_D^2+(x_D-1)^2))+4(x_C-1)x_Cy_D^2((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)(y_C(y_D+(x_D-1)y_C)+(x_C-1)x_C(x_D-1))-4(x_C-1)x_Cx_Dy_D((y_C^2+(x_C-1)x_C)y_D-(x_D-1)y_C)((y_C^2+(x_C-1)x_C)y_D+x_Dy_C)))$.

O has coordinates

$(((y_C^2+x_C(x_C+1))((2x_C-1)y_C^2+2(x_C-1)^2x_C)y_D^2-y_C(2x_Dy_C^4-y_C^4+4x_C^2x_Dy_C^2-2x_C^2y_C^2+2x_C^4x_D-4x_C^2x_D+2x_Cx_D-x_C^4+x_C^2)y_D+(x_D-1)x_Dy_C^4+(x_C-1)x_C(x_D-1)x_Dy_C^2)/(2(((3x_C^2-3x_C+1)y_C^2+3x_C^4-6x_C^3+3x_C^2)y_D^2-y_C((2x_C-1)(2x_D-1)y_C^2+2(x_C-1)x_C((2x_C-1)x_D-x_C))y_D+(x_D-1)x_Dy_C^4+(x_C-1)x_C(x_D-1)x_Dy_C^2)), (y_C((y_C^2-2(x_C-1)x_C)y_D^2-y_C(y_C^2-2(2x_C-1)x_D+x_C(x_C+1))y_D-(x_D-1)x_Dy_C^2))/(2(((3x_C^2-3x_C+1)y_C^2+3x_C^4-6x_C^3+3x_C^2)y_D^2-y_C((2x_C-1)(2x_D-1)y_C^2+2(x_C-1)x_C((2x_C-1)x_D-x_C))y_D+(x_D-1)x_Dy_C^4+(x_C-1)x_C(x_D-1)x_Dy_C^2)))$.

Line DNO is given by the equation

$y=(((y_C((y_C^2-2(x_C-1)x_C)y_D^2-y_C(y_C^2-2(2x_C-1)x_D+x_C(x_C+1))y_D+(1-x_D)x_Dy_C^2))/(2(((3x_C^2-3x_C+1)y_C^2+3x_C^4-6x_C^3+3x_C^2)y_D^2-y_C((2x_C-1)(2x_D-1)y_C^2+2(x_C-1)x_C((2x_C-1)x_D-x_C))y_D+(x_D-1)x_Dy_C^4+(x_C-1)x_C(x_D-1)x_Dy_C^2))-y_D)(x-x_D))/(((y_C^2+x_C(x_C+1))((2x_C-1)y_C^2+2(x_C-1)^2x_C)y_D^2-y_C(2x_Dy_C^4-y_C^4+4x_C^2x_Dy_C^2-2x_C^2y_C^2+2x_C^4x_D-4x_C^2x_D+2x_Cx_D-x_C^4+x_C^2)y_D+(x_D-1)x_Dy_C^4+(x_C-1)x_C(x_D-1)x_Dy_C^2)/(2(((3x_C^2-3x_C+1)y_C^2+3x_C^4-6x_C^3+3x_C^2)y_D^2-y_C((2x_C-1)(2x_D-1)y_C^2+2(x_C-1)x_C((2x_C-1)x_D-x_C))y_D+(x_D-1)x_Dy_C^4+(x_C-1)x_C(x_D-1)x_Dy_C^2))-x_D)+y_D$.

δ is given by the equation

$y=((((x_C*y_D-x_D*y_C)*(y_C*(y_D-y_C)+(x_C-1)*(x_D-x_C)))/(y_C*((y_D-y_C)^2+(x_D-x_C)^2))-1/2)*(x-((x_C*y_D-x_D*y_C)*(y_C*(y_D-y_C)+(x_C-1)*(x_D-x_C)))/(y_C*((y_D-y_C)^2+(x_D-x_C)^2))))/((y_C^2+(x_C-1)*x_C)/(2*y_C)-(((x_C-1)*y_D-(x_D-1)*y_C)*(x_C*y_D-x_D*y_C))/(y_C*((y_D-y_C)^2+(x_D-x_C)^2)))+(((x_C-1)*y_D-(x_D-1)*y_C)*(x_C*y_D-x_D*y_C))/(y_C*((y_D-y_C)^2+(x_D-x_C)^2))$.

ε is given by the equation

$y=(((y_D*(y_C*(y_D+(x_D-1)*y_C)+(x_C-1)*x_C*(x_D-1)))/(y_C*(y_D^2+(x_D-1)^2))-1/2)*(x-(y_D*(y_C*(y_D+(x_D-1)*y_C)+(x_C-1)*x_C*(x_D-1)))/(y_C*(y_D^2+(x_D-1)^2))))/((y_C^2+(x_C-1)*x_C)/(2*y_C)-(y_D*((y_C^2+(x_C-1)*x_C)*y_D-(x_D-1)*y_C))/(y_C*(y_D^2+(x_D-1)^2)))+(y_D*((y_C^2+(x_C-1)*x_C)*y_D-(x_D-1)*y_C))/(y_C*(y_D^2+(x_D-1)^2))$.

ζ is given by the equation

$y=(((x_D*((y_C^2+(x_C-1)*x_C)*y_D+x_D*y_C))/(y_C*(y_D^2+x_D^2))-1/2)*(x-(x_D*((y_C^2+(x_C-1)*x_C)*y_D+x_D*y_C))/(y_C*(y_D^2+x_D^2))))/((y_C^2+(x_C-1)*x_C)/(2*y_C)-(y_D*((y_C^2+(x_C-1)*x_C)*y_D+x_D*y_C))/(y_C*(y_D^2+x_D^2)))+(y_D*((y_C^2+(x_C-1)*x_C)*y_D+x_D*y_C))/(y_C*(y_D^2+x_D^2))$.

P has coordinates

$(-(-x_C^4*y_D+2*x_C^3*x_D*y_D-x_C^2*x_D^2*y_C+x_C^2*x_D*y_C-2*x_C^2*x_D*y_D-2*x_C^2*y_C^2*y_D+x_C^2*y_C*y_D^2+x_C^2*y_D+x_C*x_D^2*y_C+2*x_C*x_D*y_C^2*y_D-x_C*x_D*y_C+x_C*y_C*y_D^2-x_D^2*y_C^3+x_D*y_C^3-2*x_D*y_C^2*y_D-y_C^4*y_D+y_C^3*y_D^2)/(2*y_C*(x_C^2*x_D-x_C^2-x_C*x_D^2+x_C*x_D-x_C*y_D^2+x_D*y_C^2-y_C^2+y_C*y_D)), -(2*x_C^3*y_D^2-2*x_C^2*x_D*y_C*y_D+x_C^2*y_C*y_D-2*x_C^2*y_D^2+2*x_C*y_C^2*y_D^2+x_C*y_C*y_D+x_D^2*y_C^2-2*x_D*y_C^3*y_D-x_D*y_C^2+y_C^3*y_D-y_C^2*y_D^2)/(2*y_C*(x_C^2*x_D-x_C^2-x_C*x_D^2+x_C*x_D-x_C*y_D^2+x_D*y_C^2-y_C^2+y_C*y_D)))$.

Q has coordinates
$((2*x_C^4*x_D*y_D-x_C^4*y_D-4*x_C^3*x_D*y_D+2*x_C^3*y_D+x_C^2*x_D^2*y_C+4*x_C^2*x_D*y_C^2*y_D-x_C^2*x_D*y_C+2*x_C^2*x_D*y_D-2*x_C^2*y_C^2*y_D+x_C^2*y_C*y_D^2-x_C^2*y_D-x_C*x_D^2*y_C-4*x_C*x_D*y_C^2*y_D+x_C*x_D*y_C+2*x_C*y_C^2*y_D-x_C*y_C*y_D^2+x_D^2*y_C^3+2*x_D*y_C^4*y_D-x_D*y_C^3+2*x_D*y_C^2*y_D-y_C^4*y_D+y_C^3*y_D^2)/(2*y_C*(x_C^2*x_D^2-x_C^2*x_D+x_C^2*y_D^2-x_C*x_D^2+x_C*x_D-x_C*y_D^2+x_D^2*y_C^2-x_D*y_C^2+y_C^2*y_D^2+y_C*y_D)), (2*x_C^4*y_D^2-4*x_C^3*y_D^2+4*x_C^2*y_C^2*y_D^2+x_C^2*y_C*y_D+2*x_C^2*y_D^2-4*x_C*y_C^2*y_D^2-x_C*y_C*y_D-x_D^2*y_C^2+x_D*y_C^2+2*y_C^4*y_D^2+y_C^3*y_D+y_C^2*y_D^2)/(2*y_C*(x_C^2*x_D^2-x_C^2*x_D+x_C^2*y_D^2-x_C*x_D^2+x_C*x_D-x_C*y_D^2+x_D^2*y_C^2-x_D*y_C^2+y_C^2*y_D^2+y_C*y_D)))$.
R has coordinates
$(-(x_C^4*y_D-2*x_C^3*x_D*y_D-2*x_C^3*y_D+x_C^2*x_D^2*y_C+x_C^2*x_D*y_C+4*x_C^2*x_D*y_D+2*x_C^2*y_C^2*y_D-x_C^2*y_C*y_D^2+x_C^2*y_D-3*x_C*x_D^2*y_C-2*x_C*x_D*y_C^2*y_D-x_C*x_D*y_C-2*x_C*x_D*y_D-2*x_C*y_C^2*y_D+x_C*y_C*y_D^2+x_D^2*y_C^3+2*x_D^2*y_C+x_D*y_C^3+y_C^4*y_D-y_C^3*y_D^2)/(2*y_C*(-x_C^2*x_D+x_C*x_D^2+x_C*x_D+x_C*y_D^2-x_D^2-x_D*y_C^2+y_C*y_D-y_D^2)), -(-2*x_C^3*y_D^2+2*x_C^2*x_D*y_C*y_D-x_C^2*y_C*y_D+4*x_C^2*y_D^2-4*x_C*x_D*y_C*y_D-2*x_C*y_C^2*y_D^2+x_C*y_C*y_D-2*x_C*y_D^2+x_D^2*y_C^2+2*x_D*y_C^3*y_D-x_D*y_C^2+2*x_D*y_C*y_D-y_C^3*y_D+y_C^2*y_D^2)/(2*y_C*(-x_C^2*x_D+x_C*x_D^2+x_C*x_D+x_C*y_D^2-x_D^2-x_D*y_C^2+y_C*y_D-y_D^2)))$.
Line AP is given by the equation
$y=((((2*x_C-1)*y_C^2+2*x_C^3-2*x_C^2)*y_D^2+((1-2*x_D)*y_C^3+(-2*x_C^2*x_D+x_C^2+x_C)*y_C)*y_D+(x_D^2-x_D)*y_C^2)*x)/((y_C^3+(x_C^2+x_C)*y_C)*y_D^2+(-y_C^4+((2*x_C-2)*x_D-2*x_C^2)*y_C^2+(2*x_C^3-2*x_C^2)*x_D-x_C^4+x_C^2)*y_D+(x_D-x_D^2)*y_C^3+((x_C-x_C^2)*x_D^2+(x_C^2-x_C)*x_D)*y_C)$.
Line BR is given by the equation
$y=((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2-y_C*((2*x_D-1)*y_C^2+2*(x_C-1)^2*x_D-(x_C-1)*x_C)*y_D-(x_D-1)*x_D*y_C^2)*(x-1))/(y_C*(y_C^2+(x_C-2)*(x_C-1))*y_D^2-(y_C^2*(y_C^2-2*x_C*(x_D-x_C+1)+2)-(x_C-1)^2*x_C*(2*x_D-x_C))*y_D-(x_D-1)*x_D*y_C^3-(x_C-1)*x_C*(x_D-1)*x_D*y_C)$.
Line CQ is given by the equation
$y=-((((2*x_C^2-2*x_C+1)*y_C^2+2*x_C^4-4*x_C^3+2*x_C^2)*y_D^2-y_C*(y_C^2-(x_C-1)*x_C)*y_D-2*(x_D-1)*x_D*y_C^4-(2*x_C^2-2*x_C+1)*(x_D-1)*x_D*y_C^2)*x-(y_C^2+x_C^2)*((2*x_C-1)*y_C^2+2*(x_C-1)^2*x_C)*y_D^2+y_C*(y_C^2+x_C^2)*((2*x_D-1)*y_C^2+2*(x_C-1)^2*x_D-(x_C-1)*x_C)*y_D+(x_D-1)*x_D*y_C^4+x_C^2*(x_D-1)*x_D*y_C^2)/((2*x_C-1)*y_C*(y_C^2+(x_C-1)*x_C)*y_D^2-(2*x_D*y_C^4-y_C^4+4*x_C^2*x_D*y_C^2-4*x_C*x_D*y_C^2+2*x_D*y_C^2-2*x_C^2*y_C^2+2*x_C^4*x_D-4*x_C^3*x_D+2*x_C^2*x_D-x_C^4+2*x_C^3-x_C^2)*y_D+(2*x_C-1)*(x_D-1)*x_D*y_C^3+(x_C-1)*x_C*(2*x_C-1)*(x_D-1)*x_D*y_C)$.
Consider the point S whose coordinates are given by
$(((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2+((1-2*x_D)*y_C^3+((-2*x_C^2+4*x_C-2)*x_D+x_C^2-x_C)*y_C)*y_D+(x_D-x_D^2)*y_C^2)*((y_C^3+(x_C^2+x_C)*y_C)*y_D^2+(-y_C^4+((2*x_C-2)*x_D-2*x_C^2)*y_C^2+(2*x_C^3-2*x_C^2)*x_D-x_C^4+x_C^2)*y_D+(x_D-x_D^2)*y_C^3+((x_C-x_C^2)*x_D^2+(x_C^2-x_C)*x_D)*y_C))/(2*(((3*x_C^2-3*x_C+1)*y_C^3+(3*x_C^4-6*x_C^3+3*x_C^2)*y_C)*y_D^4+(((2-4*x_C)*x_D-x_C^2+3*x_C-2)*y_C^4+((-4*x_C^3+6*x_C^2-2*x_C)*x_D-2*x_C^4+6*x_C^3-3*x_C^2-x_C)*y_C^2-x_C^6+3*x_C^5-3*x_C^4+x_C^3)*y_D^3+((x_D^2-x_D+1)*y_C^5+((4*x_C^2-4*x_C+1)*x_D^2+(1-4*x_C^2)*x_D+x_C^2+x_C)*y_C^3+((3*x_C^4-6*x_C^3+3*x_C^2)*x_D^2+(-3*x_C^4+6*x_C^3-3*x_C^2)*x_D)*y_C)*y_D^2+((x_D^2-x_D)*y_C^6+((2-4*x_C)*x_D^3+(2*x_C^2+4*x_C-1)*x_D^2+(-2*x_C^2-1)*x_D)*y_C^4+((-4*x_C^3+6*x_C^2-2*x_C)*x_D^3+(x_C^4+4*x_C^3-7*x_C^2+2*x_C)*x_D^2+(x_C^2-x_C^4)*x_D)*y_C^2)*y_D+(x_D^4-2*x_D^3+x_D^2)*y_C^5+((x_C^2-x_C)*x_D^4+(2*x_C-2*x_C^2)*x_D^3+(x_C^2-x_C)*x_D^2)*y_C^3)), ((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2-y_C*((2*x_D-1)*y_C^2+2*(x_C-1)^2*x_D-(x_C-1)*x_C)*y_D-(x_D-1)*x_D*y_C^2)*((2*x_C-$

$$1)*y_C^2+2*x_C^3-2*x_C^2)*y_D^2-y_C*((2*x_D-1)*y_C^2+x_C*(x_C*(2*x_D-1)-1))*y_D+(x_D-1)*x_D*y_C^2))/(2*(y_C*((3*x_C^2-3*x_C+1)*y_C^2+3*(x_C-1)^2*x_C^2)*y_D^4-(4*x_C*x_D*y_C^4-2*x_D*y_C^4+x_C^2*y_C^4-3*x_C*y_C^4+2*y_C^4+4*x_C^3*x_D*y_C^2-6*x_C^2*x_D*y_C^2+2*x_C*x_D*y_C^2+2*x_C^4*y_C^2-6*x_C^3*y_C^2+3*x_C^2*y_C^2+x_C*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)*y_D^3+y_C*(x_D^2*y_C^4-x_D*y_C^4+y_C^4+4*x_C^2*x_D^2*y_C^2-4*x_C*x_D^2*y_C^2+x_D^2*y_C^2-4*x_C^2*x_D*y_C^2+x_D*y_C^2+x_C^2*y_C^2+x_C*y_C^2+3*x_C^4*x_D^2-6*x_C^3*x_D^2+3*x_C^2*x_D^2-3*x_C^4*x_D+6*x_C^3*x_D-3*x_C^2*x_D)*y_D^2+(x_D-1)*x_D*y_C^2*(y_C^2*(y_C^2-2*((2*x_C-1)*x_D-x_C^2)+1)-(x_C-1)*x_C*(2*(2*x_C-1)*x_D-x_C*(x_C+1)))*y_D+(x_D-1)^2*x_D^2*y_C^5+(x_C-1)*x_C*(x_D-1)^2*x_D^2*y_C^3)))$$

We verify that S lies on lines AP, BR, and CQ since we calculate

$$((((2*x_C-1)*y_C^2+2*x_C^3-2*x_C^2)*y_D^2+((1-2*x_D)*y_C^3+(-2*x_C^2*x_D+x_C^2+x_C)*y_C)*y_D+(x_D^2-x_D)*y_C^2)*(((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2+((1-2*x_D)*y_C^3+((-2*x_C^2+4*x_C-2)*x_D+x_C^2-x_C)*y_C)*y_D+(x_D-x_D^2)*y_C^2)*((y_C^3+(x_C^2+x_C)*y_C)*y_D^2+(-y_C^4+((2*x_C-2)*x_D-2*x_C^2)*y_C^2+(2*x_C^3-2*x_C^2)*x_D-x_C^4+x_C^2)*y_D+(x_D-x_D^2)*y_C^3+((x_C-x_C^2)*x_D^2+(x_C^2-x_C)*x_D)*y_C))/(2*(((3*x_C^2-3*x_C+1)*y_C^3+(3*x_C^4-6*x_C^3+3*x_C^2)*y_C)*y_D^4+(((2-4*x_C)*x_D-x_C^2+3*x_C-2)*y_C^4+((-4*x_C^3+6*x_C^2-2*x_C)*x_D-2*x_C^4+6*x_C^3-3*x_C^2-x_C)*y_C^2-x_C^6+3*x_C^5-3*x_C^4+x_C^3)*y_D^3+((x_D^2-x_D+1)*y_C^5+((4*x_C^2-4*x_C+1)*x_D^2+(1-4*x_C^2)*x_D+x_C^2+x_C)*y_C^3+((3*x_C^4-6*x_C^3+3*x_C^2)*x_D^2+(-3*x_C^4+6*x_C^3-3*x_C^2)*x_D)*y_C)*y_D^2+((x_D^2-x_D)*y_C^6+((2-4*x_C)*x_D^3+(2*x_C^2+4*x_C-1)*x_D^2+(-2*x_C^2-1)*x_D)*y_C^4+((-4*x_C^3+6*x_C^2-2*x_C)*x_D^3+(x_C^4+4*x_C^3-7*x_C^2+2*x_C)*x_D^2+(x_C^2-x_C^4)*x_D)*y_C^2)*y_D+(x_D^4-2*x_D^3+x_D^2)*y_C^5+((x_C^2-x_C)*x_D^4+(2*x_C-2*x_C^2)*x_D^3+(x_C^2-x_C)*x_D^2)*y_C^3))))/((y_C^3+(x_C^2+x_C)*y_C)*y_D^2+(-y_C^4+((2*x_C-2)*x_D-2*x_C^2)*y_C^2+(2*x_C^3-2*x_C^2)*x_D-x_C^4+x_C^2)*y_D+(x_D-x_D^2)*y_C^3+((x_C-x_C^2)*x_D^2+(x_C^2-x_C)*x_D)*y_C)-((((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2-y_C*((2*x_D-1)*y_C^2+2*(x_C-1)^2*x_D-(x_C-1)*x_C)*y_D-(x_D-1)*x_D*y_C^2)*(((2*x_C-1)*y_C^2+2*x_C^3-2*x_C^2)*y_D^2-y_C*((2*x_D-1)*y_C^2+x_C*(x_C*(2*x_D-1)-1))*y_D+(x_D-1)*x_D*y_C^2))/(2*(y_C*((3*x_C^2-3*x_C+1)*y_C^2+3*(x_C-1)^2*x_C^2)*y_D^4-(4*x_C*x_D*y_C^4-2*x_D*y_C^4+x_C^2*y_C^4-3*x_C*y_C^4+2*y_C^4+4*x_C^3*x_D*y_C^2-6*x_C^2*x_D*y_C^2+2*x_C*x_D*y_C^2+2*x_C^4*y_C^2-6*x_C^3*y_C^2+3*x_C^2*y_C^2+x_C*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)*y_D^3+y_C*(x_D^2*y_C^4-x_D*y_C^4+y_C^4+4*x_C^2*x_D^2*y_C^2-4*x_C*x_D^2*y_C^2+x_D^2*y_C^2-4*x_C^2*x_D*y_C^2+x_D*y_C^2+x_C^2*y_C^2+x_C*y_C^2+3*x_C^4*x_D^2-6*x_C^3*x_D^2+3*x_C^2*x_D^2-3*x_C^4*x_D+6*x_C^3*x_D-3*x_C^2*x_D)*y_D^2+(x_D-1)*x_D*y_C^2*(y_C^2*(y_C^2-2*((2*x_C-1)*x_D-x_C^2)+1)-(x_C-1)*x_C*(2*(2*x_C-1)*x_D-x_C*(x_C+1)))*y_D+(x_D-1)^2*x_D^2*y_C^5+(x_C-1)*x_C*(x_D-1)^2*x_D^2*y_C^3))) = 0,$$

$$-((((2*x_C^2-2*x_C+1)*y_C^2+2*x_C^4-4*x_C^3+2*x_C^2)*y_D^2-y_C*(y_C^2-(x_C-1)*x_C)*y_D-2*(x_D-1)*x_D*y_C^4-(2*x_C^2-2*x_C+1)*(x_D-1)*x_D*y_C^2)*(((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2+((1-2*x_D)*y_C^3+((-2*x_C^2+4*x_C-2)*x_D+x_C^2-x_C)*y_C)*y_D+(x_D-x_D^2)*y_C^2)*((y_C^3+(x_C^2+x_C)*y_C)*y_D^2+(-y_C^4+((2*x_C-2)*x_D-2*x_C^2)*y_C^2+(2*x_C^3-2*x_C^2)*x_D-x_C^4+x_C^2)*y_D+(x_D-x_D^2)*y_C^3+((x_C-x_C^2)*x_D^2+(x_C^2-x_C)*x_D)*y_C))/(2*(((3*x_C^2-3*x_C+1)*y_C^3+(3*x_C^4-6*x_C^3+3*x_C^2)*y_C)*y_D^4+(((2-4*x_C)*x_D-x_C^2+3*x_C-2)*y_C^4+((-4*x_C^3+6*x_C^2-2*x_C)*x_D-2*x_C^4+6*x_C^3-3*x_C^2-x_C)*y_C^2-x_C^6+3*x_C^5-3*x_C^4+x_C^3)*y_D^3+((x_D^2-x_D+1)*y_C^5+((4*x_C^2-4*x_C+1)*x_D^2+(1-4*x_C^2)*x_D+x_C^2+x_C)*y_C^3+((3*x_C^4-6*x_C^3+3*x_C^2)*x_D^2+(-3*x_C^4+6*x_C^3-3*x_C^2)*x_D)*y_C)*y_D^2+((x_D^2-x_D)*y_C^6+((2-4*x_C)*x_D^3+(2*x_C^2+4*x_C-1)*x_D^2+(-2*x_C^2-1)*x_D)*y_C^4+((-4*x_C^3+6*x_C^2-2*x_C)*x_D^3+(x_C^4+4*x_C^3-7*x_C^2+2*x_C)*x_D^2+(x_C^2-x_C^4)*x_D)*y_C^2)*y_D+(x_D^4-2*x_D^3+x_D^2)*y_C^5+((x_C^2-x_C)*x_D^4+(2*x_C-2*x_C^2)*x_D^3+(x_C^2-x_C)*x_D^2)*y_C^3)))-(y_C^2+x_C^2)*((2*x_C-1)*y_C^2+2*(x_C-1)^2*x_C)*y_D^2+y_C*(y_C^2+x_C^2)*((2*x_D-1)*y_C^2+2*(x_C-1)^2*x_D-(x_C-1)*x_C)*y_D+(x_D-1)*x_D*y_C^4+x_C^2*(x_D-1)*x_D*y_C^2)/((2*x_C-1)*y_C*(y_C^2+(x_C-1)*x_C)*y_D^2-(2*x_D*y_C^4-y_C^4+4*x_C^2*x_D*y_C^2-4*x_C*x_D*y_C^2+2*x_D*y_C^2-2*x_C^2*y_C^2+2*x_C^4*x_D-4*x_C^3*x_D+2*x_C^2*x_D-x_C^4+2*x_C^3-x_C^2)*y_D+(2*x_C-1)*(x_D-1)*x_D*y_C^3+(x_C-1)*x_C*(2*x_C-1)*(x_D-$$

$1)*x_D*y_C)-((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2-y_C*((2*x_D-1)*y_C^2+2*(x_C-1)^2*x_D-(x_C-1)*x_C)*y_D-(x_D-1)*x_D*y_C^2)*(((2*x_C-1)*y_C^2+2*x_C^3-2*x_C^2)*y_D^2-y_C*((2*x_D-1)*y_C^2+x_C*(x_C*(2*x_D-1)-1))*y_D+(x_D-1)*x_D*y_C^2))/(2*(y_C*((3*x_C^2-3*x_C+1)*y_C^2+3*(x_C-1)^2*x_C^2)*y_D^4-(4*x_C*x_D*y_C^4-2*x_D*y_C^4+x_C^2*y_C^4-3*x_C*y_C^4+2*y_C^4+4*x_C^3*x_D*y_C^2-6*x_C^2*x_D*y_C^2+2*x_C*x_D*y_C^2+2*x_C^4*y_C^2-6*x_C^3*y_C^2+3*x_C^2*y_C^2+x_C*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)*y_D^3+y_C*(x_D^2*y_C^4-x_D*y_C^4+y_C^4+4*x_C^2*x_D^2*y_C^2-4*x_C*x_D^2*y_C^2+x_D^2*y_C^2-4*x_C^2*x_D*y_C^2+x_D*y_C^2+x_C^2*y_C^2+x_C*y_C^2+3*x_C^4*x_D^2-6*x_C^3*x_D^2+3*x_C^2*x_D^2-3*x_C^4*x_D+6*x_C^3*x_D-3*x_C^2*x_D)*y_D^2+(x_D-1)*x_D*y_C^2*(y_C^2*(y_C^2-2*((2*x_C-1)*x_D-x_C^2)+1)-(x_C-1)*x_C*(2*(2*x_C-1)*x_D-x_C*(x_C+1)))*y_D+(x_D-1)^2*x_D^2*y_C^5+(x_C-1)*x_C*(x_D-1)^2*x_D^2*y_C^3)) = 0,$

and

$((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2-y_C*((2*x_D-1)*y_C^2+2*(x_C-1)^2*x_D-(x_C-1)*x_C)*y_D-(x_D-1)*x_D*y_C^2)*((((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2+((1-2*x_D)*y_C^3+((-2*x_C^2+4*x_C-2)*x_D+x_C^2-x_C)*y_C)*y_D+(x_D-x_D^2)*y_C^2)*((y_C^3+(x_C^2+x_C)*y_C)*y_D^2+(-y_C^4+((2*x_C-2)*x_D-2*x_C^2)*y_C^2+(2*x_C^3-2*x_C^2)*x_D-x_C^4+x_C^2)*y_D+(x_D-x_D^2)*y_C^3+((x_C-x_C^2)*x_D^2+(x_C^2-x_C)*x_D)*y_C))/(2*(((3*x_C^2-3*x_C+1)*y_C^3+(3*x_C^4-6*x_C^3+3*x_C^2)*y_C)*y_D^4+(((2-4*x_C)*x_D-x_C^2+3*x_C-2)*y_C^4+((-4*x_C^3+6*x_C^2-2*x_C)*x_D-2*x_C^4+6*x_C^3-3*x_C^2-x_C)*y_C^2-x_C^6+3*x_C^5-3*x_C^4+x_C^3)*y_D^3+((x_D^2-x_D+1)*y_C^5+((4*x_C^2-4*x_C+1)*x_D^2+(1-4*x_C^2)*x_D+x_C^2+x_C)*y_C^3+((3*x_C^4-6*x_C^3+3*x_C^2)*x_D^2+(-3*x_C^4+6*x_C^3-3*x_C^2)*x_D)*y_C)*y_D^2+((x_D^2-x_D)*y_C^6+((2-4*x_C)*x_D^3+(2*x_C^2+4*x_C-1)*x_D^2+(-2*x_C^2-1)*x_D)*y_C^4+((-4*x_C^3+6*x_C^2-2*x_C)*x_D^3+(x_C^4+4*x_C^3-7*x_C^2+2*x_C)*x_D^2+(x_C^2-x_C^4)*x_D)*y_C^2)*y_D+(x_D^4-2*x_D^3+x_D^2)*y_C^5+((x_C^2-x_C)*x_D^4+(2*x_C-2*x_C^2)*x_D^3+(x_C^2-x_C)*x_D^2)*y_C^3)))-1))/(y_C*(y_C^2+(x_C-2)*(x_C-1))*y_D^2-(y_C^2*(y_C^2-2*x_C*(x_D-x_C+1)+2)-(x_C-1)^2*x_C*(2*x_D-x_C))*y_D-(x_D-1)*x_D*y_C^3-(x_C-1)*x_C*(x_D-1)*x_D*y_C)-((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2-y_C*((2*x_D-1)*y_C^2+2*(x_C-1)^2*x_D-(x_C-1)*x_C)*y_D-(x_D-1)*x_D*y_C^2)*(((2*x_C-1)*y_C^2+2*x_C^3-2*x_C^2)*y_D^2-y_C*((2*x_D-1)*y_C^2+x_C*(x_C*(2*x_D-1)-1))*y_D+(x_D-1)*x_D*y_C^2))/(2*(y_C*((3*x_C^2-3*x_C+1)*y_C^2+3*(x_C-1)^2*x_C^2)*y_D^4-(4*x_C*x_D*y_C^4-2*x_D*y_C^4+x_C^2*y_C^4-3*x_C*y_C^4+2*y_C^4+4*x_C^3*x_D*y_C^2-6*x_C^2*x_D*y_C^2+2*x_C*x_D*y_C^2+2*x_C^4*y_C^2-6*x_C^3*y_C^2+3*x_C^2*y_C^2+x_C*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)*y_D^3+y_C*(x_D^2*y_C^4-x_D*y_C^4+y_C^4+4*x_C^2*x_D^2*y_C^2-4*x_C*x_D^2*y_C^2+x_D^2*y_C^2-4*x_C^2*x_D*y_C^2+x_D*y_C^2+x_C^2*y_C^2+x_C*y_C^2+3*x_C^4*x_D^2-6*x_C^3*x_D^2+3*x_C^2*x_D^2-3*x_C^4*x_D+6*x_C^3*x_D-3*x_C^2*x_D)*y_D^2+(x_D-1)*x_D*y_C^2*(y_C^2*(y_C^2-2*((2*x_C-1)*x_D-x_C^2)+1)-(x_C-1)*x_C*(2*(2*x_C-1)*x_D-x_C*(x_C+1)))*y_D+(x_D-1)^2*x_D^2*y_C^5+(x_C-1)*x_C*(x_D-1)^2*x_D^2*y_C^3))) = 0.$

We verify that S lies on line DNO since we calculate that

$(((y_C((y_C^2-2(x_C-1)x_C)y_D^2-y_C(y_C^2-2(2x_C-1)x_D+x_C(x_C+1))y_D+(1-x_D)x_Dy_C^2))/(2(((3x_C^2-3x_C+1)y_C^2+3x_C^4-6x_C^3+3x_C^2)y_D^2-y_C((2x_C-1)(2x_D-1)y_C^2+2(x_C-1)x_C((2x_C-1)x_D-x_C))y_D+(x_D-1)x_Dy_C^4+(x_C-1)x_C(x_D-1)x_Dy_C^2))-y_D)((((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2+((1-2*x_D)*y_C^3+((-2*x_C^2+4*x_C-2)*x_D+x_C^2-x_C)*y_C)*y_D+(x_D-x_D^2)*y_C^2)*((y_C^3+(x_C^2+x_C)*y_C)*y_D^2+(-y_C^4+((2*x_C-2)*x_D-2*x_C^2)*y_C^2+(2*x_C^3-2*x_C^2)*x_D-x_C^4+x_C^2)*y_D+(x_D-x_D^2)*y_C^3+((x_C-x_C^2)*x_D^2+(x_C^2-x_C)*x_D)*y_C))/(2*(((3*x_C^2-3*x_C+1)*y_C^3+(3*x_C^4-6*x_C^3+3*x_C^2)*y_C)*y_D^4+(((2-4*x_C)*x_D-x_C^2+3*x_C-2)*y_C^4+((-4*x_C^3+6*x_C^2-2*x_C)*x_D-2*x_C^4+6*x_C^3-3*x_C^2-x_C)*y_C^2-x_C^6+3*x_C^5-3*x_C^4+x_C^3)*y_D^3+((x_D^2-x_D+1)*y_C^5+((4*x_C^2-4*x_C+1)*x_D^2+(1-4*x_C^2)*x_D+x_C^2+x_C)*y_C^3+((3*x_C^4-6*x_C^3+3*x_C^2)*x_D^2+(-3*x_C^4+6*x_C^3-3*x_C^2)*x_D)*y_C)*y_D^2+((x_D^2-x_D)*y_C^6+((2-4*x_C)*x_D^3+(2*x_C^2+4*x_C-1)*x_D^2+(-2*x_C^2-1)*x_D)*y_C^4+((-4*x_C^3+6*x_C^2-2*x_C)*x_D^3+(x_C^4+4*x_C^3-7*x_C^2+2*x_C)*x_D^2+(x_C^2-x_C^4)*x_D)*y_C^2)*y_D+(x_D^4-2*x_D^3+x_D^2)*y_C^5+$

$((x_C^2-x_C)*x_D^4+(2*x_C-2*x_C^2)*x_D^3+(x_C^2-x_C)*x_D^2)*y_C^3)))-x_D))/(((y_C^2+x_C(x_C+1))((2x_C-1)y_C^2+2(x_C-1)^2x_C)y_D^2-y_C(2x_Dy_C^4-y_C^4+4x_C^2x_Dy_C^2-2x_C^2y_C^2+2x_C^4x_D-4x_C^2x_D+2x_Cx_D-x_C^4+x_C^2)y_D+(x_D-1)x_Dy_C^4+(x_C-1)x_C(x_D-1)x_Dy_C^2)/(2(((3x_C^2-3x_C+1)y_C^2+3x_C^4-6x_C^3+3x_C^2)y_D^2-y_C((2x_C-1)(2x_D-1)y_C^2+2(x_C-1)x_C((2x_C-1)x_D-x_C))y_D+(x_D-1)x_Dy_C^4+(x_C-1)x_C(x_D-1)x_Dy_C^2))-x_D)+y_D-(((((2*x_C-1)*y_C^2+2*x_C^3-4*x_C^2+2*x_C)*y_D^2-y_C*((2*x_D-1)*y_C^2+2*(x_C-1)^2*x_D-(x_C-1)*x_C)*y_D-(x_D-1)*x_D*y_C^2)*(((2*x_C-1)*y_C^2+2*x_C^3-2*x_C^2)*y_D^2-y_C*((2*x_D-1)*y_C^2+x_C*(x_C*(2*x_D-1)-1))*y_D+(x_D-1)*x_D*y_C^2))/(2*(y_C*((3*x_C^2-3*x_C+1)*y_C^2+3*(x_C-1)^2*x_C^2)*y_D^4-(4*x_C*x_D*y_C^4-2*x_D*y_C^4+x_C^2*y_C^4-3*x_C*y_C^4+2*y_C^4+4*x_C^3*x_D*y_C^2-6*x_C^2*x_D*y_C^2+2*x_C*x_D*y_C^2+2*x_C^4*y_C^2-6*x_C^3*y_C^2+3*x_C^2*y_C^2+x_C*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)*y_D^3+y_C*(x_D^2*y_C^4-x_D*y_C^4+y_C^4+4*x_C^2*x_D^2*y_C^2-4*x_C*x_D^2*y_C^2+x_D^2*y_C^2-4*x_C^2*x_D*y_C^2+x_D*y_C^2+x_C^2*y_C^2+x_C*y_C^2+3*x_C^4*x_D^2-6*x_C^3*x_D^2+3*x_C^2*x_D^2-3*x_C^4*x_D+6*x_C^3*x_D-3*x_C^2*x_D)*y_D^2+(x_D-1)*x_D*y_C^2*(y_C^2*(y_C^2-2*((2*x_C-1)*x_D-x_C^2)+1)-(x_C-1)*x_C*(2*(2*x_C-1)*x_D-x_C*(x_C+1)))*y_D+(x_D-1)^2*x_D^2*y_C^5+(x_C-1)*x_C*(x_D-1)^2*x_D^2*y_C^3))) = 0$
as desired. □

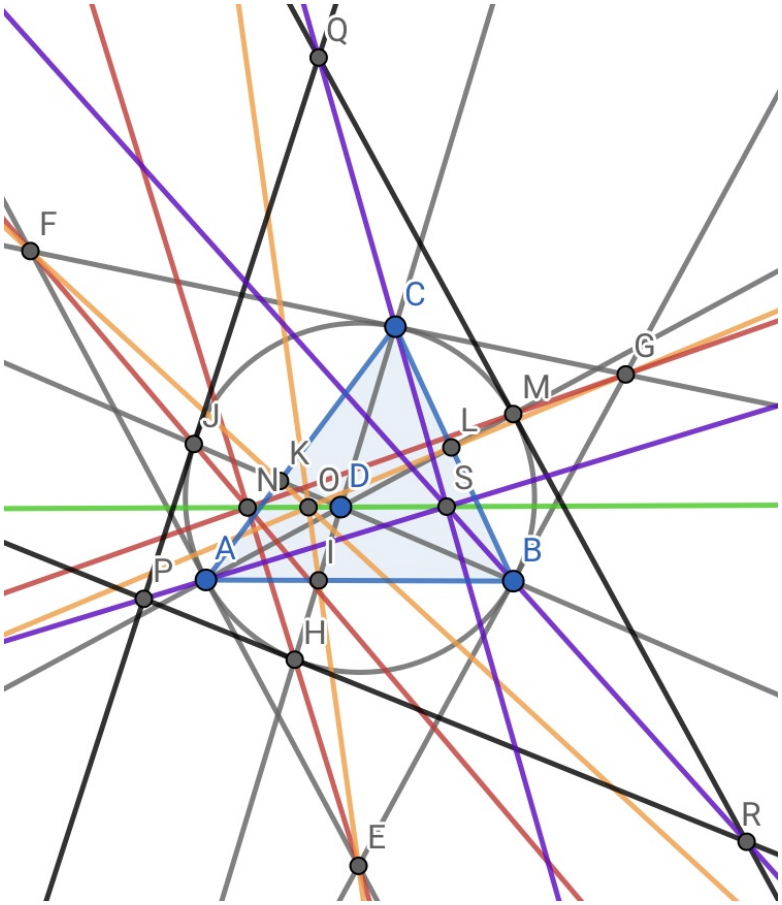

Figure 1. Illustration.

References

[1] Warren, B.L., "Generalization of the Circumcenter, Exeter Point, and Euler Line", *International Journal of Computer Discovered Mathematics*, Under Review.