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Generalization of Van Aubel's Theorem

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1. Results

Theorem 1.1. *Let A, B, C, and D be the vertices of a given quadrilateral.*
Let E be some point in the plane.
Let points A, D, F, and G form a square with G adjacent to A.
Let points C, D, H, and I form a square with H adjacent to C.
Let points B, C, J, and K form a square with J adjacent to B.
Let points A, B, L, and M form a square with L adjacent to A.
Let N be a point on line AE.
Let O be the point on line DE such that $DO : EO = AN : EN$.
Let P be the point on line EF such that $FP : EP = AN : EN$.
Let Q be the point on line EG such that $GQ : EQ = AN : EN$.
Let lines AP, DQ, FN, and GO concur at point R.
Let S be the point on line CE such that $CS : ES = AN : EN$.
Let T be the point on line EH such that $HT : ET = AN : EN$.
Let U be the point on line EI such that $IU : EU = AN : EN$.
Let lines CU, DT, HO, and IS concur at point V.
Let W be the point on line BE such that $BW : EW = AN : EN$.
Let Z be the point on line JE such that $JZ : EZ = AN : EN$.
Let A_1 be the point on line EK such that $KA_1 : EA_1 = AN : EN$.
Let lines BA_1, CZ, JS, and KW concur at point B_1.
Let C_1 be the point on line EL such that $LC_1 : EC_1 = AN : EN$.
Let D_1 be the point on line EM such that $MD_1 : ED_1 = AN : EN$.
Let lines AD_1, BC_1, LW, and MN concur at point E_1.

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Let lines RB_1 and VE_1 concur at point F_1 if the squares are external to the quadrilateral.
Let lines RB_1 and VE_1 concur at point G_1 if the squares are internal to the quadrilateral.
Let H_1 be the midpoint of R and B_1.
Let I_1 be the midpoint of V and E_1.
Then,
1. line segments RB_1 and VE_1 are of equal length,
2. lines RB_1 and VE_1 are orthogonal, and
3. points F_1, G_1, H_1, and I_1 are concyclic.

Proof. Without loss of generality, we will use Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates (x_E, y_E).

Case 1: All the squares are external. We prove only this case.
F has coordinates $(x_D - y_D, x_D + y_D)$.
G has coordinates $(-y_D, x_D)$.
H has coordinates $(x_C - y_C + y_D, x_C - x_D + y_C)$.
I has coordinates $(x_D - y_C + y_D, x_C - x_D + y_D)$.
J has coordinates $(y_C + 1, 1 - x_C)$.
K has coordinates $(x_C + y_C, y_C - x_C + 1)$.
L has coordinates $(0, -1)$.
M has coordinates $(1, -1)$.
N has coordinates $(x_N, (x_N * y_E)/x_E)$.
O has coordinates $(((x_E - x_D) * x_N)/x_E + x_D, (x_N * y_E + (x_E - x_N) * y_D)/x_E)$.
P has coordinates
$((x_N*(y_D+x_E-x_D))/x_E-y_D+x_D, (x_N*y_E+(x_E-x_N)*y_D-x_D*x_N+x_D*x_E)/x_E)$.
Q has coordinates $((x_N * (y_D + x_E))/x_E - y_D, (x_N * y_E - x_D * x_N + x_D * x_E)/x_E)$.
R has coordinates
$((x_D * x_E - x_D * x_N + x_E * x_N - x_E * y_D + x_N * y_D)/(2 * x_E - x_N), (x_D * x_E - x_D * x_N + x_E * y_D - x_N * y_D + x_N * y_E)/(2 * x_E - x_N))$.
S has coordinates
$(((x_E - x_C) * x_N)/x_E + x_C, (x_N * y_E + (x_E - x_N) * y_C)/x_E)$.
T has coordinates
$(y_D + (x_N * (-y_D + y_C + x_E - x_C))/x_E - y_C + x_C, (x_N * y_E + (x_E - x_N) * y_C + (x_D - x_C) * x_N + (x_C - x_D) * x_E)/x_E)$.
U has coordinates
$(y_D + (x_N * (-y_D + y_C + x_E - x_D))/x_E - y_C + x_D, (x_N * y_E + (x_E - x_N) * y_D + (x_D - x_C) * x_N + (x_C - x_D) * x_E)/x_E)$.
V has coordinates $((x_C * x_E + x_D * x_E - x_C * x_N - x_D * x_N + x_E * x_N - x_E * y_C + x_E * y_D + x_N * y_C - x_N * y_D)/(2 * x_E - x_N), (x_C * x_E - x_D * x_E - x_C * x_N + x_D * x_N + x_E * y_C + x_E * y_D - x_N * y_C - x_N * y_D + x_N * y_E)/(2 * x_E - x_N))$.
W has coordinates $(((x_E - 1) * x_N)/x_E + 1, (x_N * y_E)/x_E)$.
Z has coordinates $(y_C+(x_N*(-y_C+x_E-1))/x_E+1, (x_N*(y_E+x_C-1))/x_E-x_C+1)$.
A_1 has coordinates

$(y_C+(x_N*(-y_C+x_E-x_C))/x_E+x_C, (x_N*(y_E-y_C+x_C-1))/x_E+y_C-x_C+1)$.
B_1 has coordinates
$((x_E-x_N+x_C*x_E-x_C*x_N+x_E*x_N+x_E*y_C-x_N*y_C)/(2*x_E-x_N), (x_E-x_N-x_C*x_E+x_C*x_N+x_E*y_C-x_N*y_C+x_N*y_E)/(2*x_E-x_N))$.
C_1 has coordinates $(x_N, (x_N*y_E+x_N-x_E)/x_E)$.
D_1 has coordinates $(((x_E-1)*x_N)/x_E+1, (x_N*y_E+x_N-x_E)/x_E)$.
E_1 has coordinates
$((x_E-x_N+x_E*x_N)/(2*x_E-x_N), (x_N-x_E+x_N*y_E)/(2*x_E-x_N))$.
Line RB_1 is given by the equation
$y=-((x_N-2*x_E)*(y_D-y_C+x_D+x_C-1)*x+x_N*(y_D+y_C-x_D+x_C+1)*y_E-(2*x_N*y_C-2*x_E*y_C-x_E*x_N+2*x_N-2*x_E)*y_D-x_E*x_N*y_C+((x_D+x_C-1)*x_E-2*x_C*x_D)*x_N+2*x_C*x_D*x_E)/((x_N-2*x_E)*(y_D+y_C-x_D+x_C+1))$.
Line VE_1 is given by the equation
$y=((x_N-2*x_E)*(y_D+y_C-x_D+x_C+1)*x-x_N*(y_D-y_C+x_D+x_C-1)*y_E+((x_E-2)*x_N+2*x_E)*y_D+x_E*x_N*y_C-((x_D-x_C-1)*x_E+2*x_C)*x_N+2*x_C*x_E)/((x_N-2*x_E)*(y_D-y_C+x_D+x_C-1))$.
F_1 has coordinates
$((((x_N-x_E)*y_C-(x_E-2)*x_N-2*x_E)*y_D^2-(x_N*y_C^2-x_E*y_C^2-x_D*x_N*y_C-x_C*x_N*y_C+x_N*y_C+x_D*x_E*y_C+x_C*x_E*y_C-x_E*y_C+2*x_C*x_E*x_N-x_C*x_D*x_N-3*x_C*x_N+x_C*x_D*x_E+3*x_C*x_E)*y_D-x_E*x_N*y_C^2+(x_D-1)*(2*x_E*x_N-x_C*x_N+x_C*x_E)*y_C-(((x_D-1)^2+x_C^2)*x_E-x_C*(x_D*(x_D+x_C-2)+x_C+1))*x_N-x_C*(x_D*(x_D+x_C-2)+x_C+1)*x_E)/((x_N-2*x_E)*((y_D+x_C)^2+(y_C-x_D)*(y_C-x_D+2)+1)), -(x_N*((y_D+x_C)^2+(y_C-x_D+1)^2)*y_E-(x_N-x_E)*y_C*y_D^2-(x_N-x_E)*(y_C*(y_C-x_D+x_C+3)+(x_C-2)*x_D-x_C+2)*y_D-x_C*(x_D+1)*(x_N-x_E)*y_C+x_C*(x_D-1)*(x_D-x_C+1)*x_N-x_C*(x_D-1)*(x_D-x_C+1)*x_E)/((x_N-2*x_E)*((y_D+x_C)^2+(y_C-x_D)*(y_C-x_D+2)+1)))$.
H_1 has coordinates
$(-((x_N-x_E)*y_D+(x_E-x_N)*y_C+(2*x_E-x_D-x_C-1)*x_N+(x_D+x_C+1)*x_E)/(2*x_N-4*x_E), -(x_N*(2*y_E-y_D-y_C-x_D+x_C-1)+x_E*(y_D+y_C+x_D-x_C+1))/(2*(x_N-2*x_E)))$.
I_1 has coordinates
$(((x_N-x_E)*y_D+(x_E-x_N)*y_C+(-2*x_E+x_D+x_C+1)*x_N+(-x_D-x_C-1)*x_E)/(2*x_N-4*x_E), -(x_N*(2*y_E-y_D-y_C+x_D-x_C+1)+x_E*(y_D+y_C-x_D+x_C-1))/(2*(x_N-2*x_E)))$.
We verify that line segments RB_1 and VE_1 are of equal length since we calculate
$(x_R-x_{B_1})^2+(y_R-y_{B_1})^2-((x_V-x_{E_1})^2+(y_V-y_{E_1})^2)$
$=(((x_D*x_E-x_D*x_N+x_E*x_N-x_E*y_D+x_N*y_D)/(2*x_E-x_N))-((x_E-x_N+x_C*x_E-x_C*x_N+x_E*x_N+x_E*y_C-x_N*y_C)/(2*x_E-x_N)))^2+(((x_D*x_E-x_D*x_N+x_E*y_D-x_N*y_D+x_N*y_E)/(2*x_E-x_N))-((x_E-x_N-x_C*x_E+x_C*x_N+x_E*y_C-x_N*y_C+x_N*y_E)/(2*x_E-x_N)))^2-((((x_C*x_E+x_D*x_E-x_C*x_N-x_D*x_N+x_E*x_N-x_E*y_C+x_E*y_D+x_N*y_C-x_N*y_D)/(2*x_E-x_N))-((x_E-x_N+x_E*x_N)/(2*x_E-x_N)))^2+(((x_C*x_E-x_D*x_E-x_C*x_N+x_D*x_N+x_E*y_C+x_E*y_D-x_N*y_C-x_N*y_D+x_N*y_E)/(2*x_E-x_N))-((x_N-x_E+x_N*y_E)/(2*x_E-x_N)))^2)$
$=0$ as desired.
We next verify that lines RB_1 and VE_1 are orthogonal since we calculate
$((y_R-y_{B_1})/(x_R-x_{B_1}))*((y_V-y_{E_1})/(x_V-x_{E_1}))$
$=((((x_D*x_E-x_D*x_N+x_E*y_D-x_N*y_D+x_N*y_E)/(2*x_E-x_N))-((x_E-x_N-x_C*x_E+x_C*x_N+x_E*y_C-x_N*y_C+x_N*y_E)/(2*x_E-x_N)))/(((x_D*x_E-x_D*x_N+x_E*x_N-x_E*y_D+x_N*y_D)/(2*x_E-x_N))-((x_E-x_N+x_C*x_E-x_C*x_N+$

$x_E * x_N + x_E * y_C - x_N * y_C)/(2 * x_E - x_N)))) * ((((x_C * x_E - x_D * x_E - x_C * x_N + x_D * x_N + x_E * y_C + x_E * y_D - x_N * y_C - x_N * y_D + x_N * y_E)/(2 * x_E - x_N)) - ((x_N - x_E + x_N * y_E)/(2 * x_E - x_N)))/(((x_C * x_E + x_D * x_E - x_C * x_N - x_D * x_N + x_E * x_N - x_E * y_C + x_E * y_D + x_N * y_C - x_N * y_D)/(2 * x_E - x_N)) - ((x_E - x_N + x_E * x_N)/(2 * x_E - x_N))))$ $= -1$ as desired.

□

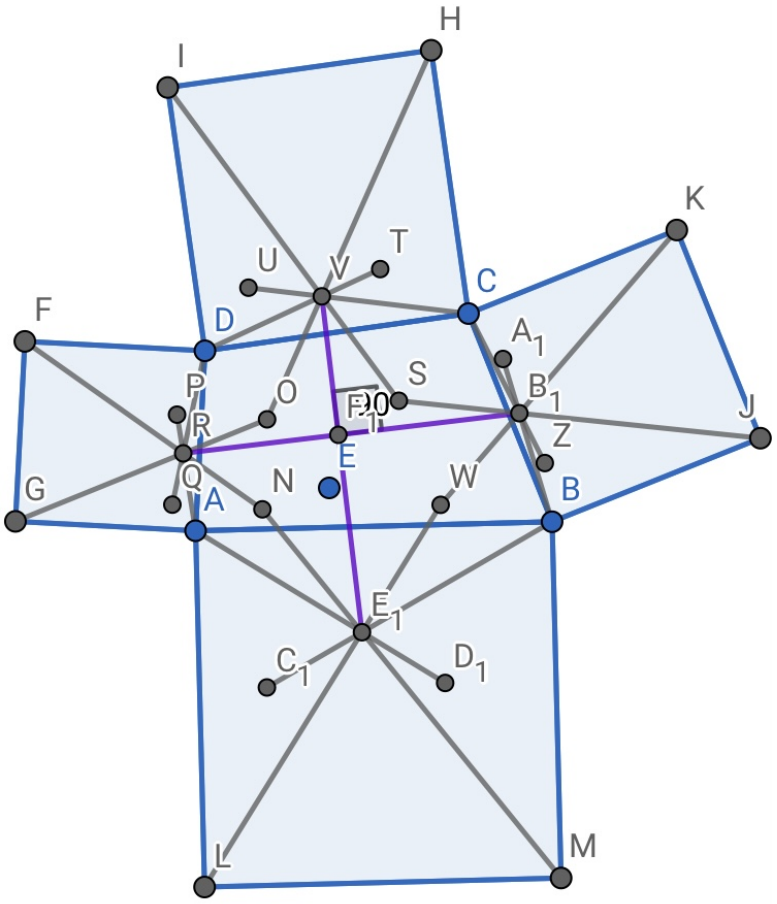

FIGURE 1. Lengths RB_1 and VE_1 are equal and lines RB_1 and VE_1 are orthogonal.

REFERENCES

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