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Three Concurrent Circles in the Context of a Certain Type of Hexagon

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1. Introduction

If a hexagon has the property that each of the reflections of three non-adjacent vertices about the line formed by the adjacent vertices yields a single point, then the three circles, each of which passes through three adjacent vertices at a time, are concurrent. We give a proof of an equivalent formulation of this here.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be a point in the plane.
Let E be the reflection of D about line AB.
Let F be the reflection of D about line AC.
Let G be the reflection of D about line BC.
Then,
1. Circles ABE, ACF, and BCG are concurrent, and
2. Circles AEF, BEG, and CFG are concurrent.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates $(x_D, -y_D)$.
F has coordinates

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$(2*(x_C*(x_C*x_D+y_C*y_D))/(x_C^2+y_C^2)-x_D, 2*(y_C*(x_C*x_D+y_C*y_D))/(x_C^2+y_C^2)-y_D)$.
G has coordinates
$(2*(y_C*((x_C-1)*y_D+y_C)+(x_C-1)^2*x_D)/(y_C^2+(x_C-1)^2)-x_D, 2*(y_C*(y_C*y_D+(x_C-1)*x_D-x_C+1))/(y_C^2+(x_C-1)^2)-y_D)$.
1. Circle ABE is given by the equation
$(x-1/2)^2+(y-(-(y_D^2+(x_D-1)*x_D)/(2*y_D)))^2-1/4-(-(y_D^2+(x_D-1)*x_D)/(2*y_D))^2=0$.
Circle ACF is given by the equation
$(x-(y_C*y_D^2-(y_C-x_C)*(y_C+x_C)*y_D+x_D*(x_D-2*x_C)*y_C)/(2*(x_C*y_D-x_D*y_C)))^2+(y-(-(x_C*y_D^2-2*x_C*y_C*y_D+x_D*y_C^2+x_C*x_D^2-x_C^2*x_D)/(2*(x_C*y_D-x_D*y_C))))^2-((y_C*y_D^2-(y_C-x_C)*(y_C+x_C)*y_D+x_D*(x_D-2*x_C)*y_C)/(2*(x_C*y_D-x_D*y_C)))^2-((-(x_C*y_D^2-2*x_C*y_C*y_D+x_D*y_C^2+x_C*x_D^2-x_C^2*x_D)/(2*(x_C*y_D-x_D*y_C))))^2=0$.
Circle BCG is given by the equation
$(x-(y_C*y_D^2-(y_C^2-x_C^2+1)*y_D+(x_D-1)*(x_D-2*x_C-1)*y_C)/(2*((x_C-1)*y_D-(x_D-1)*y_C)))^2+(y-(-((x_C-1)*y_D^2+(2-2*x_C)*y_C*y_D+(x_D-1)*y_C^2+(x_C-1)*x_D^2+(1-x_C^2)*x_D+x_C^2-x_C)/(2*((x_C-1)*y_D+(1-x_D)*y_C))))^2-(1-(y_C*y_D^2-(y_C^2-x_C^2+1)*y_D+(x_D-1)*(x_D-2*x_C-1)*y_C)/(2*((x_C-1)*y_D-(x_D-1)*y_C)))^2-(-(-((x_C-1)*y_D^2+(2-2*x_C)*y_C*y_D+(x_D-1)*y_C^2+(x_C-1)*x_D^2+(1-x_C^2)*x_D+x_C^2-x_C)/(2*((x_C-1)*y_D+(1-x_D)*y_C))))^2=0$.
Consider the point H whose coordinates are given by
$(((x_D*y_D*(-y_C^2+x_C*(x_C-1))-x_D^2*y_C*(x_D-1)+y_C*y_D^2*(2*x_C-x_D))*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D-y_C*(2*x_C-x_D)*(x_D-1)))/(y_C^2*y_D^4+y_D^2*(y_C^2*(2*x_D*(x_D-4*x_C+1)+2*x_C*(x_C+1))+x_C^2-2*x_C^3+x_C^4+y_C^4)+x_D^2*y_C^2*(x_D-1)^2+2*y_C*y_D^3*(-y_C^2+x_C*(x_C-1))-x_D*y_C*y_D*(2*x_D-2)*(-y_C^2+x_C*(x_C-1))), -(y_D*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D+x_D*y_C*(x_D-2*x_C+1))*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D-y_C*(2*x_C-x_D)*(x_D-1)))/(y_C^2*y_D^4+y_D^2*(y_C^2*(2*x_D*(x_D-4*x_C+1)+2*x_C*(x_C+1))+x_C^2-2*x_C^3+x_C^4+y_C^4)+x_D^2*y_C^2*(x_D-1)^2+2*y_C*y_D^3*(-y_C^2+x_C*(x_C-1))-x_D*y_C*y_D*(2*x_D-2)*(-y_C^2+x_C*(x_C-1))))$.
We verify that H lies on circles ABE, ACF, and BCG since we calculate
$(((x_D*y_D*(-y_C^2+x_C*(x_C-1))-x_D^2*y_C*(x_D-1)+y_C*y_D^2*(2*x_C-x_D))*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D-y_C*(2*x_C-x_D)*(x_D-1)))/(y_C^2*y_D^4+y_D^2*(y_C^2*(2*x_D*(x_D-4*x_C+1)+2*x_C*(x_C+1))+x_C^2-2*x_C^3+x_C^4+y_C^4)+x_D^2*y_C^2*(x_D-1)^2+2*y_C*y_D^3*(-y_C^2+x_C*(x_C-1))-x_D*y_C*y_D*(2*x_D-2)*(-y_C^2+x_C*(x_C-1)))-1/2)^2+(-(y_D*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D+x_D*y_C*(x_D-2*x_C+1))*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D-y_C*(2*x_C-x_D)*(x_D-1)))/(y_C^2*y_D^4+y_D^2*(y_C^2*(2*x_D*(x_D-4*x_C+1)+2*x_C*(x_C+1))+x_C^2-2*x_C^3+x_C^4+y_C^4)+x_D^2*y_C^2*(x_D-1)^2+2*y_C*y_D^3*(-y_C^2+x_C*(x_C-1))-x_D*y_C*y_D*(2*x_D-2)*(-y_C^2+x_C*(x_C-1)))-(-(y_D^2+(x_D-1)*x_D)/(2*y_D)))^2-1/4-(-(y_D^2+(x_D-1)*x_D)/(2*y_D))^2=0$,
$(((x_D*y_D*(-y_C^2+x_C*(x_C-1))-x_D^2*y_C*(x_D-1)+y_C*y_D^2*(2*x_C-x_D))*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D-y_C*(2*x_C-x_D)*(x_D-1)))/(y_C^2*y_D^4+y_D^2*(y_C^2*(2*x_D*(x_D-4*x_C+1)+2*x_C*(x_C+1))+x_C^2-2*x_C^3+x_C^4+y_C^4)+x_D^2*y_C^2*(x_D-1)^2+2*y_C*y_D^3*(-y_C^2+x_C*(x_C-1))-x_D*y_C*y_D*(2*x_D-2)*(-y_C^2+x_C*(x_C-1)))-(y_C*y_D^2-(y_C^2-x_C^2+1)*y_D+(x_D-1)*(x_D-2*x_C-1)*y_C)/(2*((x_C-1)*y_D-(x_D-1)*y_C)))^2+(-(y_D*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D+x_D*y_C*(x_D-2*x_C+1))*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D-y_C*(2*x_C-x_D)*(x_D-1)))/(y_C^2*y_D^4+y_D^2*(y_C^2*(2*x_D*(x_D-4*x_C+1)+2*x_C*(x_C+1))+x_C^2-2*x_C^3+x_C^4+y_C^4)+x_D^2*y_C^2*(x_D-1)^2+2*y_C*y_D^3*(-y_C^2+x_C*(x_C-1))-x_D*y_C*y_D*(2*x_D-2)*(-y_C^2+x_C*(x_C-1)))-((x_C-1)*y_D^2+(2-$

$2*x_C)*y_C*y_D+(x_D-1)*y_C^2+(x_C-1)*x_D^2+(1-x_C^2)*x_D+x_C^2-x_C)/(2*((x_C-1)*y_D+(1-x_D)*y_C))))^2-(1-(y_C*y_D^2-(y_C^2-x_C^2+1)*y_D+(x_D-1)*(x_D-2*x_C-1)*y_C)/(2*((x_C-1)*y_D-(x_D-1)*y_C)))^2-(-(-((x_C-1)*y_D^2+(2-2*x_C)*y_C*y_D+(x_D-1)*y_C^2+(x_C-1)*x_D^2+(1-x_C^2)*x_D+x_C^2-x_C)/(2*((x_C-1)*y_D+(1-x_D)*y_C))))^2=0,$
and
$(((x_D*y_D*(-y_C^2+x_C*(x_C-1))-x_D^2*y_C*(x_D-1)+y_C*y_D^2*(2*x_C-x_D))*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D-y_C*(2*x_C-x_D)*(x_D-1)))/(y_C^2*y_D^4+y_D^2*(y_C^2*(2*x_D*(x_D-4*x_C+1)+2*x_C*(x_C+1))+x_C^2-2*x_C^3+x_C^4+y_C^4)+x_D^2*y_C^2*(x_D-1)^2+2*y_C*y_D^3*(-y_C^2+x_C*(x_C-1))-x_D*y_C*y_D*(2*x_D-2)*(-y_C^2+x_C*(x_C-1)))-(y_C*y_D^2-(y_C^2-x_C^2+1)*y_D+(x_D-1)*(x_D-2*x_C-1)*y_C)/(2*((x_C-1)*y_D-(x_D-1)*y_C)))^2+(-(y_D*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D+x_D*y_C*(x_D-2*x_C+1))*(y_C*y_D^2+(x_C*(x_C-1)-y_C^2)*y_D-y_C*(2*x_C-x_D)*(x_D-1)))/(y_C^2*y_D^4+y_D^2*(y_C^2*(2*x_D*(x_D-4*x_C+1)+2*x_C*(x_C+1))+x_C^2-2*x_C^3+x_C^4+y_C^4)+x_D^2*y_C^2*(x_D-1)^2+2*y_C*y_D^3*(-y_C^2+x_C*(x_C-1))-x_D*y_C*y_D*(2*x_D-2)*(-y_C^2+x_C*(x_C-1)))-(-((x_C-1)*y_D^2+(2-2*x_C)*y_C*y_D+(x_D-1)*y_C^2+(x_C-1)*x_D^2+(1-x_C^2)*x_D+x_C^2-x_C)/(2*((x_C-1)*y_D+(1-x_D)*y_C))))^2-(1-(y_C*y_D^2-(y_C^2-x_C^2+1)*y_D+(x_D-1)*(x_D-2*x_C-1)*y_C)/(2*((x_C-1)*y_D-(x_D-1)*y_C)))^2-(-(-((x_C-1)*y_D^2+(2-2*x_C)*y_C*y_D+(x_D-1)*y_C^2+(x_C-1)*x_D^2+(1-x_C^2)*x_D+x_C^2-x_C)/(2*((x_C-1)*y_D+(1-x_D)*y_C))))^2=0$
as desired.

2. Circle AEF is given by the equation
$(x-(y_C*y_D+x_C*x_D)/(2*x_C))^2+(y-((x_D*y_C)/(2*x_C)-y_D/2))^2-(-(y_C*y_D+x_C*x_D)/(2*x_C))^2-(-((x_D*y_C)/(2*x_C)-y_D/2))^2=0.$
Circle BEG is given by the equation
$(x-(y_C*y_D+(x_C-1)*x_D+x_C-1)/(2*(x_C-1)))^2+(y-(((x_D-1)*y_C)/(2*(x_C-1))-y_D/2))^2-(1-(y_C*y_D+(x_C-1)*x_D+x_C-1)/(2*(x_C-1)))^2-(-(((x_D-1)*y_C)/(2*(x_C-1))-y_D/2))^2=0.$
Circle CFG is given by the equation
$(x-(y_C*((2*x_C-1)*y_D-(x_D-x_C-1)*y_C)+(x_C-1)*x_C*(x_D+x_C))/(2*(y_C^2+(x_C-1)*x_C)))^2+(y-((y_C^2-(x_C-1)*x_C)*y_D+y_C^3+((2*x_C-1)*x_D+x_C^2-2*x_C)*y_C)/(2*(y_C^2+(x_C-1)*x_C)))^2-(x_C-(y_C*((2*x_C-1)*y_D-(x_D-x_C-1)*y_C)+(x_C-1)*x_C*(x_D+x_C))/(2*(y_C^2+(x_C-1)*x_C)))^2-(y_C-((y_C^2-(x_C-1)*x_C)*y_D+y_C^3+((2*x_C-1)*x_D+x_C^2-2*x_C)*y_C)/(2*(y_C^2+(x_C-1)*x_C)))^2=0.$
Consider the point I whose coordinates are given by
$(((y_C*y_D+(x_C-1)*x_D)*(x_C*y_C*y_D-(x_D-x_C)*y_C^2+x_C^3-x_C^2))/(y_C^2*y_D^2+2*(x_C-1)*x_C*y_C*y_D+(x_D-x_C)^2*y_C^2+x_C^4-2*x_C^3+x_C^2), (y_C*(y_C*y_D+x_C*(x_D-1))*(y_C*y_D+(x_C-1)*x_D))/(y_C^2*y_D^2+2*(x_C-1)*x_C*y_C*y_D+(x_D-x_C)^2*y_C^2+x_C^4-2*x_C^3+x_C^2)).$
We verify that I lies on circles AEF, BEG, and CFG since we calculate
$(((y_C*y_D+(x_C-1)*x_D)*(x_C*y_C*y_D-(x_D-x_C)*y_C^2+x_C^3-x_C^2))/(y_C^2*y_D^2+2*(x_C-1)*x_C*y_C*y_D+(x_D-x_C)^2*y_C^2+x_C^4-2*x_C^3+x_C^2)-(y_C*y_D+x_C*x_D)/(2*x_C))^2+((y_C*(y_C*y_D+x_C*(x_D-1))*(y_C*y_D+(x_C-1)*x_D))/(y_C^2*y_D^2+2*(x_C-1)*x_C*y_C*y_D+(x_D-x_C)^2*y_C^2+x_C^4-2*x_C^3+x_C^2)-((x_D*y_C)/(2*x_C)-y_D/2))^2-(-(y_C*y_D+x_C*x_D)/(2*x_C))^2-(-((x_D*y_C)/(2*x_C)-y_D/2))^2=0,$
$(((y_C*y_D+(x_C-1)*x_D)*(x_C*y_C*y_D-(x_D-x_C)*y_C^2+x_C^3-x_C^2))/(y_C^2*y_D^2+2*(x_C-1)*x_C*y_C*y_D+(x_D-x_C)^2*y_C^2+x_C^4-2*x_C^3+x_C^2)-(y_C*y_D+(x_C-1)*x_D+x_C-1)/(2*(x_C-1)))^2+((y_C*(y_C*y_D+x_C*(x_D-1))*(y_C*y_D+(x_C-1)*x_D))/(y_C^2*y_D^2+2*(x_C-1)*x_C*y_C*y_D+(x_D-x_C)^2*y_C^2+x_C^4-2*x_C^3+x_C^2)-(((x_D-1)*y_C)/(2*(x_C-1))-y_D/2))^2-(1-(y_C*y_D+(x_C-1)*x_D+x_C-1)/(2*(x_C-1)))^2-(-(((x_D-1)*y_C)/(2*(x_C-1))-y_D/2))^2=0,$

and
$(((y_C*y_D+(x_C-1)*x_D)*(x_C*y_C*y_D-(x_D-x_C)*y_C^2+x_C^3-x_C^2))/(y_C^2*y_D^2+2*(x_C-1)*x_C*y_C*y_D+(x_D-x_C)^2*y_C^2+x_C^4-2*x_C^3+x_C^2)-(y_C*((2*x_C-1)*y_D-(x_D-x_C-1)*y_C)+(x_C-1)*x_C*(x_D+x_C))/(2*(y_C^2+(x_C-1)*x_C)))^2+((y_C*(y_C*y_D+x_C*(x_D-1))*(y_C*y_D+(x_C-1)*x_D))/(y_C^2*y_D^2+2*(x_C-1)*x_C*y_C*y_D+(x_D-x_C)^2*y_C^2+x_C^4-2*x_C^3+x_C^2)-((y_C^2-(x_C-1)*x_C)*y_D+y_C^3+((2*x_C-1)*x_D+x_C^2-2*x_C)*y_C)/(2*(y_C^2+(x_C-1)*x_C)))^2-(x_C-(y_C*((2*x_C-1)*y_D-(x_D-x_C-1)*y_C)+(x_C-1)*x_C*(x_D+x_C))/(2*(y_C^2+(x_C-1)*x_C)))^2-(y_C-((y_C^2-(x_C-1)*x_C)*y_D+y_C^3+((2*x_C-1)*x_D+x_C^2-2*x_C)*y_C)/(2*(y_C^2+(x_C-1)*x_C)))^2=0$
as desired. □

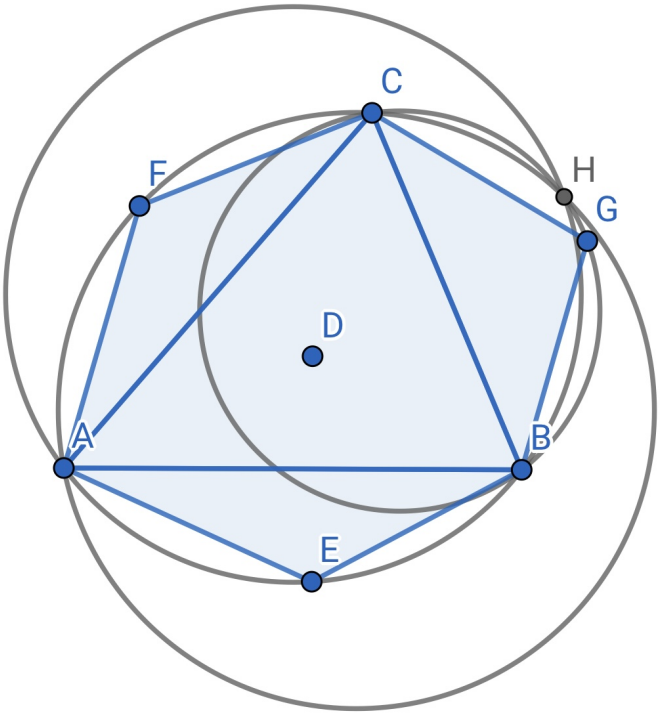

FIGURE 1. Illustration.

REFERENCES

[1] Warren, B.L., "A Circle Centered at the Nine Point Center", *International Journal of Computer Discovered Mathematics*, Under Review.

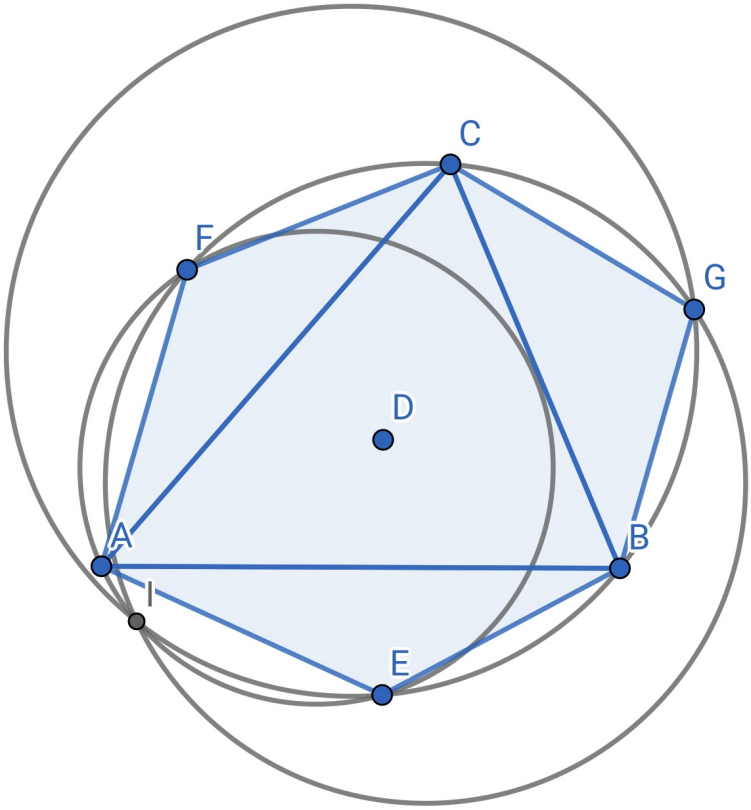

Figure 2. Illustration.