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A Class of Special Tetrahedra

Dan Ismailescu[a] [2] and Yunkyu James Lee[b]
[a] Mathematics Department, Hofstra University, Hempstead, NY 11549, USA
email: dan.p.ismailescu@hofstra.edu
[b] Columbia University, New York, NY 10027, USA.
e-mail: yl4650@columbia.edu

Abstract. We characterize the tetrahedra $A_1A_2A_3A_4$ for which

$$[A_1A_2A_3]^2 + [A_1A_3A_4]^2 = [A_1A_2A_4]^2 + [A_2A_3A_4]^2,$$

where $[A_iA_jA_k]$ denotes the area of the triangle $A_iA_jA_k$. This settles a problem raised by N. Sarlak.

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1. Introduction

A *Schläfli d-orthoscheme* is a d-dimensional simplex $A_1A_2A_3\cdots A_{d+1}$, in which the vectors $\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \ldots \overrightarrow{A_{d-1}A_d}$, and $\overrightarrow{A_dA_{d+1}}$, are mutually orthogonal vectors.

Obviously, a 2-orthoscheme is simply a right triangle, while a 3-orthoscheme is a tetrahedron all whose faces are right triangles - see figure 1.

Such a tetrahedron is also known as a *birectangular tetrahedron*, because it has two right angles at two of its vertices, as well as a *quadrirectangular tetrahedron*, because it contains four right angles (see [3] for more details).

In a very recent paper that appeared in this journal, Nina Sarlak [1] conjectured that in every 3-orthoscheme the sum of the squares of the areas of two of the faces is equal to the sum of the squares of the areas of the other two faces. She provided some numerical evidence that supported this conjecture but a rigorous proof was missing. The main goal of this paper is to find a simple characterization of the tetrahedra which satisfy this property. Sarlak's conjecture is going to follow as a simple corollary.

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[2]Corresponding author

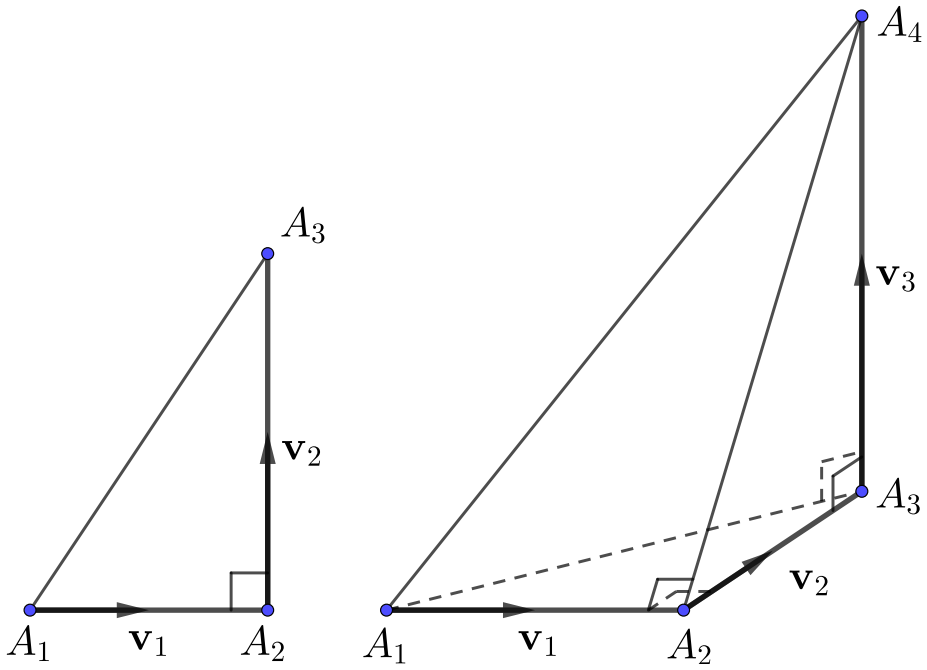

FIGURE 1. A 2-orthoscheme and a 3-orthoscheme

Theorem 1.1. *Let $A_1A_2A_3A_4$ be an arbitrary tetrahedron and let $[A_iA_jA_k]$ denote the area of the face $A_iA_jA_k$. Then, the following two statements are equivalent:*

$$[A_1A_2A_3]^2 + [A_1A_3A_4]^2 = [A_1A_2A_4]^2 + [A_2A_3A_4]^2, \tag{1}$$

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \cdot (\|\mathbf{v}_3\|^2 + \langle \mathbf{v}_1, \mathbf{v}_3 \rangle) = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle \cdot (\|\mathbf{v}_1\|^2 + \langle \mathbf{v}_1, \mathbf{v}_3 \rangle), \tag{2}$$

where $\mathbf{v}_1 = \overrightarrow{A_1A_2}, \mathbf{v}_2 = \overrightarrow{A_2A_3}, \mathbf{v}_3 = \overrightarrow{A_3A_4}$, and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle$ denotes the dot product of the vectors $\mathbf{v}_i$ and $\mathbf{v}_j$.

Proof. To simplify the computations, let us introduce the following notations:

$$\|\mathbf{v}_1\| = a, \|\mathbf{v}_2\| = b, \|\mathbf{v}_3\| = c, \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = x, \langle \mathbf{v}_1, \mathbf{v}_3 \rangle = y, \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = z.$$

Then, the left hand term in equation (1) can be written as

$$\begin{aligned} &4[A_1A_2A_3]^2 + 4[A_1A_3A_4]^2 = \|\mathbf{v}_1 \times \mathbf{v}_2\|^2 + \|(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{v}_3\|^2 = \\ &= \|\mathbf{v}_1\|^2\|\mathbf{v}_2\|^2 - \langle \mathbf{v}_1, \mathbf{v}_2 \rangle^2 + \|\mathbf{v}_1 + \mathbf{v}_2\|^2\|\mathbf{v}_3\|^2 - \langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_3 \rangle^2 = \\ &= a^2b^2 - x^2 + (a^2 + b^2 + 2x)c^2 - (y + z)^2 = \\ &= (a^2b^2 + a^2c^2 + b^2c^2 - x^2 - y^2 - z^2) + 2xc^2 - 2yz. \end{aligned} \tag{3}$$

Similar computations for the right hand term in equation (1) becomes

$$\begin{aligned} &4[A_1A_2A_4]^2 + 4[A_2A_3A_4]^2 = \|\mathbf{v}_1 \times (\mathbf{v}_2 + \mathbf{v}_3)\|^2 + \|\mathbf{v}_2 \times \mathbf{v}_3\|^2 = \\ &= \|\mathbf{v}_1\|^2\|\mathbf{v}_2 + \mathbf{v}_3\|^2 - \langle \mathbf{v}_1, \mathbf{v}_2 + \mathbf{v}_3 \rangle^2 + \|\mathbf{v}_2\|^2\|\mathbf{v}_3\|^2 - \langle \mathbf{v}_2, \mathbf{v}_3 \rangle^2 = \\ &= a^2(b^2 + c^2 + 2z) - (x + y)^2 + b^2c^2 - z^2 = \\ &= (a^2b^2 + a^2c^2 + b^2c^2 - x^2 - y^2 - z^2) + 2za^2 - 2xy. \end{aligned} \tag{4}$$

Using equalities (3) and (4), it follows that equality (1) is equivalent to

$$2xc^2 - 2yz = 2za^2 - 2xy \iff x(c^2+y) = z(a^2+y) \iff$$
$$\iff \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \cdot (\|\mathbf{v}_3\|^2 + \langle \mathbf{v}_1, \mathbf{v}_3 \rangle) = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle \cdot (\|\mathbf{v}_1\|^2 + \langle \mathbf{v}_1, \mathbf{v}_3 \rangle),$$

which is exactly what we intended to prove. □

Note that if $\overrightarrow{A_1A_2} \perp \overrightarrow{A_2A_3}$ and $\overrightarrow{A_2A_3} \perp \overrightarrow{A_3A_4}$ then equality (2) (and as a result equality (1)) is satisfied as $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0$. In particular, equality (1) is valid if $A_1A_2A_3A_4$ is the 3-orthoscheme shown in figure 1. This proves Sarlak's [1] conjecture.

A *Hill tetrahedron* is a tetrahedron $A_1A_2A_3A_4$ for which $\|\overrightarrow{A_1A_2}\| = \|\overrightarrow{A_2A_3}\| = \|\overrightarrow{A_3A_4}\|$ and $\angle A_1A_2A_3 = \angle A_2A_3A_4$ (see e. g. [2]). We claim that for such a tetrahedron equality (1) is still valid.

Indeed, using the notations introduced earlier, the fact that $A_1A_2A_3A_4$ is a Hill tetrahedron implies that $a = b = c$ and $x = z$. But then $x(c^2+y) = z(a^2+y)$ and therefore both equalities (1) and (2) are satisfied.

References

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Mathematics Department, Hofstra University, Hempstead, NY 11549.
Email address: dan.p.ismailescu@hofstra.edu

Columbia University, New York, NY 10027
Email address: yl4650@columbia.edu