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Three Euler Lines that Concur in X(20147) and a Circle Centered at the Midpoint of the Circumcenter and Nine-Point Center

Benjamin Lee Warren[a] [2]
e-mail: warrenbenjamin52@yahoo.com
web: https://orcid.org/0000-0002-2829-6953

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1. Introduction

There are several known results about concurrent Euler lines [1]. Perhaps the most famous of which is Schiffler's theorem [2]. Here, we will provide a proof of another result regarding concurrent Euler lines.

2. Results

Theorem 2.1. *Let ABC be a triangle with $A'B'C'$ the medial triangle. Let A_b and A_c be the orthogonal projections of A' on b and c respectively. Define B_a, B_c, C_a, and C_b cyclically. Then the Euler lines of AB_cC_b, BA_cC_a, and CA_bB_a are concurrent at a point which is concyclic with the circumcenters of AB_cC_b, BA_cC_a, and CA_bB_a about the midpoint of the circumcenter and the nine-point center.*

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
A' has coordinates $((x_C + 1)/2, y_C/2)$.
B' has coordinates $(x_C/2, y_C/2)$.
C' has coordinates $(1/2, 0)$.
The circumcenter has coordinates $(1/2, (y_C^2 + (x_C - 1) * x_C)/(2 * y_C))$.
The nine-point center has coordinates $(x_C/2 + 1/4, (-x_C^2 + x_C + y_C^2)/(4 * y_C))$.
A_b has coordinates

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[2]Corresponding author

$((x_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2)), (y_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2)))$.
A_c has coordinates
$((x_C+1)/2, 0)$.
B_a has coordinates
$(((x_C+1)*y_C^2+(x_C-1)^2*x_C)/(2*(y_C^2+(x_C-1)^2)), (y_C*(y_C^2+(x_C-2)*(x_C-1)))/(2*(y_C^2+(x_C-1)^2)))$.
B_c has coordinates
$(x_C/2, 0)$.
C_a has coordinates
$((y_C^2+(x_C-1)^2/2)/(y_C^2+(x_C-1)^2), ((1-x_C)*y_C)/(2*(y_C^2+(x_C-1)^2)))$.
C_b has coordinates
$(x_C^2/(2*(y_C^2+x_C^2)), (x_C*y_C)/(2*(y_C^2+x_C^2)))$.
The centroid of AB_cC_b has coordinates
$((x_C/2+x_C^2/(2*(y_C^2+x_C^2)))/3, (x_C*y_C)/(6*(y_C^2+x_C^2)))$.
The circumcenter of AB_cC_b has coordinates $(x_C/4, -((x_C-1)*x_C)/(4*y_C))$.
The Euler line of AB_cC_b is given by the equation
$y = ((x_C*y_C)/(6*(y_C^2+x_C^2)))+(x-((x_C/2+x_C^2/(2*(y_C^2+x_C^2)))/3))*(((x_C*y_C)/(6*(y_C^2+x_C^2)))-(-((x_C-1)*x_C)/(4*y_C)))/(((x_C/2+x_C^2/(2*(y_C^2+x_C^2)))/3)-(x_C/4))$.
The centroid of BA_cC_a has coordinates
$((1+(x_C+1)/2+(y_C^2+(x_C-1)^2/2)/(y_C^2+(x_C-1)^2))/3, ((1-x_C)*y_C)/(6*(y_C^2+(x_C-1)^2)))$.
The circumcenter of BA_cC_a has coordinates $((x_C+3)/4, -((x_C-1)*x_C)/(4*y_C))$.
The Euler line of BA_cC_a has coordinates
$y = (((1-x_C)*y_C)/(6*(y_C^2+(x_C-1)^2))) + (x-((1+(x_C+1)/2+(y_C^2+(x_C-1)^2/2)/(y_C^2+(x_C-1)^2))/3))*((((1-x_C)*y_C)/(6*(y_C^2+(x_C-1)^2)))-(-((x_C-1)*x_C)/(4*y_C)))/(((1+(x_C+1)/2+(y_C^2+(x_C-1)^2/2)/(y_C^2+(x_C-1)^2))/3)-((x_C+3)/4))$.
The centroid of CA_bB_a has coordinates
$((x_C+(x_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2))+((x_C+1)*y_C^2+(x_C-1)^2*x_C)/(2*(y_C^2+(x_C-1)^2)))/3, (y_C+(y_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2))+(y_C*(y_C^2+(x_C-2)*(x_C-1)))/(2*(y_C^2+(x_C-1)^2)))/3)$.
The circumcenter of CA_bB_a has coordinates $(x_C, (3*y_C^2-x_C^2+x_C)/(4*y_C))$.
The Euler line of CA_bB_a has coordinates
$y = ((y_C+(y_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2))+(y_C*(y_C^2+(x_C-2)*(x_C-1)))/(2*(y_C^2+(x_C-1)^2)))/3)+(x-((x_C+(x_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2))+((x_C+1)*y_C^2+(x_C-1)^2*x_C)/(2*(y_C^2+(x_C-1)^2)))/3))*(((y_C+(y_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2))+(y_C*(y_C^2+(x_C-2)*(x_C-1)))/(2*(y_C^2+(x_C-1)^2)))/3)-((3*y_C^2-x_C^2+x_C)/(4*y_C)))/(((x_C+(x_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2))+((x_C+1)*y_C^2+(x_C-1)^2*x_C)/(2*(y_C^2+(x_C-1)^2)))/3)-x_C)$.
Consider the point $X(20147)$ whose coordinates are given by
$(-((4*x_C-3)*y_C^4+2*x_C*(2*x_C^2-8*x_C+5)*y_C^2-9*x_C^4+18*x_C^3-9*x_C^2)/(2*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2)), ((13*x_C^2-13*x_C+3)*y_C^4+2*(x_C-1)*x_C*(11*x_C^2-11*x_C+2)*y_C^2+9*x_C^6-27*x_C^5+27*x_C^4-9*x_C^3)/(2*y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2)))$.
We verify that $X(20147)$ lies on the Euler lines of AB_cC_b, BA_cC_a, and CA_bB_a since we calculate
$((x_C*y_C)/(6*(y_C^2+x_C^2)))+(-((4*x_C-3)*y_C^4+2*x_C*(2*x_C^2-8*x_C+5)*y_C^2-9*x_C^4+18*x_C^3-9*x_C^2)/(2*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-((x_C/2+x_C^2/(2*(y_C^2+x_C^2)))/3))*(((x_C*y_C)/(6*(y_C^2+x_C^2)))-(-((x_C-$

$1)*x_C)/(4*y_C)))/(((x_C/2+x_C^2/(2*(y_C^2+x_C^2)))/3)-(x_C/4))-(((13*x_C^2-13*x_C+3)*y_C^4+2*(x_C-1)*x_C*(11*x_C^2-11*x_C+2)*y_C^2+9*x_C^6-27*x_C^5+27*x_C^4-9*x_C^3)/(2*y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2)))=0$,
$(((1-x_C)*y_C)/(6*(y_C^2+(x_C-1)^2)))+(-((4*x_C-3)*y_C^4+2*x_C*(2*x_C^2-8*x_C+5)*y_C^2-9*x_C^4+18*x_C^3-9*x_C^2)/(2*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-((1+(x_C+1)/2+(y_C^2+(x_C-1)^2/2)/(y_C^2+(x_C-1)^2))/3))*((((1-x_C)*y_C)/(6*(y_C^2+(x_C-1)^2)))-(-((x_C-1)*x_C)/(4*y_C)))/(((1+(x_C+1)/2+(y_C^2+(x_C-1)^2/2)/(y_C^2+(x_C-1)^2))/3)-((x_C+3)/4))-(((13*x_C^2-13*x_C+3)*y_C^4+2*(x_C-1)*x_C*(11*x_C^2-11*x_C+2)*y_C^2+9*x_C^6-27*x_C^5+27*x_C^4-9*x_C^3)/(2*y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2)))=0$,
and
$((y_C+(y_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2))+(y_C*(y_C^2+(x_C-2)*(x_C-1)))/(2*(y_C^2+(x_C-1)^2)))/3)+(-((4*x_C-3)*y_C^4+2*x_C*(2*x_C^2-8*x_C+5)*y_C^2-9*x_C^4+18*x_C^3-9*x_C^2)/(2*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-((x_C+(x_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2))+((x_C+1)*y_C^2+(x_C-1)^2*x_C)/(2*(y_C^2+(x_C-1)^2)))/3))*(((y_C+(y_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2))+(y_C*(y_C^2+(x_C-2)*(x_C-1)))/(2*(y_C^2+(x_C-1)^2)))/3)-((3*y_C^2-x_C^2+x_C)/(4*y_C)))/(((x_C+(x_C*(y_C^2+x_C*(x_C+1)))/(2*(y_C^2+x_C^2))+((x_C+1)*y_C^2+(x_C-1)^2*x_C)/(2*(y_C^2+(x_C-1)^2)))/3)-x_C)-(((13*x_C^2-13*x_C+3)*y_C^4+2*(x_C-1)*x_C*(11*x_C^2-11*x_C+2)*y_C^2+9*x_C^6-27*x_C^5+27*x_C^4-9*x_C^3)/(2*y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2)))=0$
as desired.

Now, consider the circle centered at the midpoint of the circumcenter and the nine-point center ($X(140)$) which passes through $X(20417)$. This circle is given by the equation
$(x-(1/2+x_C/2+1/4)/2)^2+(y-((y_C^2+(x_C-1)*x_C)/(2*y_C)+(-x_C^2+x_C+y_C^2)/(4*y_C)))/2)^2-(-((4*x_C-3)*y_C^4+2*x_C*(2*x_C^2-8*x_C+5)*y_C^2-9*x_C^4+18*x_C^3-9*x_C^2)/(2*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-(1/2+x_C/2+1/4)/2)^2-(((13*x_C^2-13*x_C+3)*y_C^4+2*(x_C-1)*x_C*(11*x_C^2-11*x_C+2)*y_C^2+9*x_C^6-27*x_C^5+27*x_C^4-9*x_C^3)/(2*y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-((y_C^2+(x_C-1)*x_C)/(2*y_C)+(-x_C^2+x_C+y_C^2)/(4*y_C)))/2)^2=0$.
We verify that the circumcenters of AB_cC_b, BA_cC_a, and CA_bB_a lie on this circle since we calculate
$(x_C/4-(1/2+x_C/2+1/4)/2)^2+(-((x_C-1)*x_C)/(4*y_C)-((y_C^2+(x_C-1)*x_C)/(2*y_C)+(-x_C^2+x_C+y_C^2)/(4*y_C)))/2)^2-(-((4*x_C-3)*y_C^4+2*x_C*(2*x_C^2-8*x_C+5)*y_C^2-9*x_C^4+18*x_C^3-9*x_C^2)/(2*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-(1/2+x_C/2+1/4)/2)^2-(((13*x_C^2-13*x_C+3)*y_C^4+2*(x_C-1)*x_C*(11*x_C^2-11*x_C+2)*y_C^2+9*x_C^6-27*x_C^5+27*x_C^4-9*x_C^3)/(2*y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-((y_C^2+(x_C-1)*x_C)/(2*y_C)+(-x_C^2+x_C+y_C^2)/(4*y_C)))/2)^2=0$,
$((x_C+3)/4-(1/2+x_C/2+1/4)/2)^2+(-((x_C-1)*x_C)/(4*y_C)-((y_C^2+(x_C-1)*x_C)/(2*y_C)+(-x_C^2+x_C+y_C^2)/(4*y_C)))/2)^2-(-((4*x_C-3)*y_C^4+2*x_C*(2*x_C^2-8*x_C+5)*y_C^2-9*x_C^4+18*x_C^3-9*x_C^2)/(2*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-(1/2+x_C/2+1/4)/2)^2-(((13*x_C^2-13*x_C+3)*y_C^4+2*(x_C-1)*x_C*(11*x_C^2-11*x_C+2)*y_C^2+9*x_C^6-27*x_C^5+27*x_C^4-9*x_C^3)/(2*y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-((y_C^2+(x_C-1)*x_C)/(2*y_C)+(-x_C^2+x_C+y_C^2)/(4*y_C)))/2)^2=0$,
and
$(x_C-(1/2+x_C/2+1/4)/2)^2+((3*y_C^2-x_C^2+x_C)/(4*y_C)-((y_C^2+(x_C-1)*x_C)/(2*y_C)+(-x_C^2+x_C+y_C^2)/(4*y_C)))/2)^2-(-((4*x_C-3)*y_C^4+2*x_C*(2*x_C^2-8*x_C+5)*y_C^2-9*x_C^4+18*x_C^3-9*x_C^2)/(2*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-$

$(1/2+x_C/2+1/4)/2)^2-(((13*x_C^2-13*x_C+3)*y_C^4+2*(x_C-1)*x_C*(11*x_C^2-11*x_C+2)*y_C^2+9*x_C^6-27*x_C^5+27*x_C^4-9*x_C^3)/(2*y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-((y_C^2+(x_C-1)*x_C)/(2*y_C)+(-x_C^2+x_C+y_C^2)/(4*y_C))/2)^2=0$ as desired. □

Other properties involving this configuration can be found in Euclid 8052 and the description of $X(67302)$, such as the remark that the radius of this circle is three-quarters the length of the circumradius.

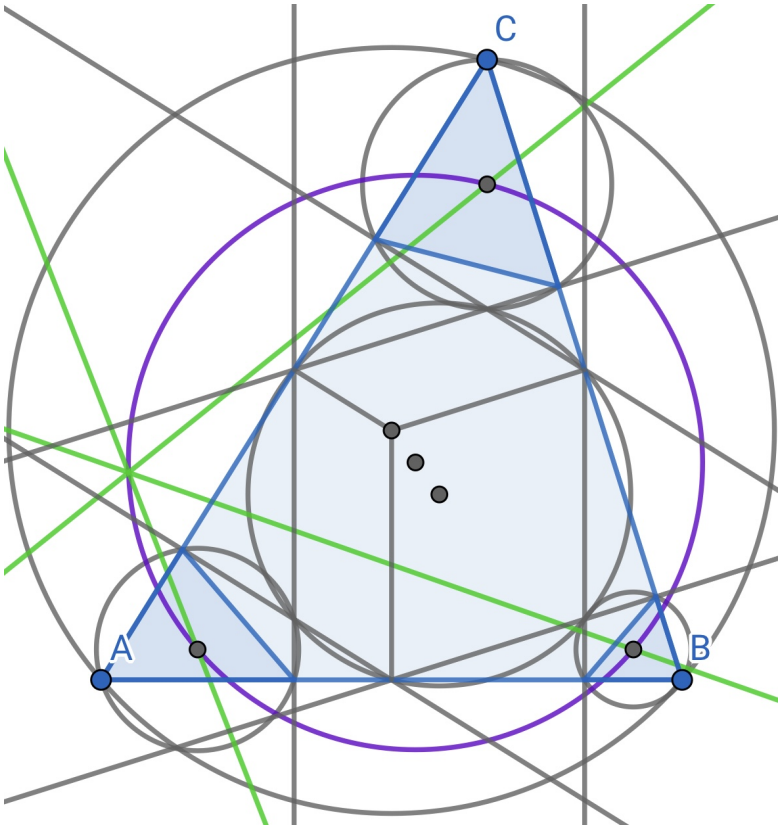

FIGURE 1. Illustration.

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