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Three More Points on the Nine-Point Circle

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1. Introduction

We pick a point on the circumcircle and demonstrate that three emergent points lie on the nine point circle.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let P be a point on the circumcircle of ABC.
Let P_A be the reflection of P about line BC.
Define P_B and P_C cyclically.
Let P'_A be the midpoint of A and P_A.
Define P'_B and P'_C cyclically.
Then P'_A, P'_B, and P'_C lie on the nine point circle of ABC.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
P temporarily has coordinates (x_P, y_P).
P_A has coordinates
$(2 * (y_C * ((x_C - 1) * y_P + y_C) + (x_C - 1)^2 * x_P)/(y_C^2 + (x_C - 1)^2) - x_P, 2 * (y_C * (y_C * y_P + (x_C - 1) * x_P - x_C + 1))/(y_C^2 + (x_C - 1)^2) - y_P)$.
P_B has coordinates
$(2*(x_C*(x_C*x_P+y_C*y_P))/(x_C^2+y_C^2)-x_P, 2*(y_C*(x_C*x_P+y_C*y_P))/(x_C^2+y_C^2)-y_P)$.
P_C has coordinates

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$(x_P, -y_P)$.
P'_A has coordinates
$((2 * (y_C * ((x_C - 1) * y_P + y_C) + (x_C - 1)^2 * x_P)/(y_C^2 + (x_C - 1)^2) - x_P)/2, (2 * (y_C * (y_C * y_P + (x_C - 1) * x_P - x_C + 1))/(y_C^2 + (x_C - 1)^2) - y_P)/2)$.
P'_B has coordinates
$((1 + 2 * (x_C * (x_C * x_P + y_C * y_P))/(x_C^2 + y_C^2) - x_P)/2, (2 * (y_C * (x_C * x_P + y_C * y_P))/(x_C^2 + y_C^2) - y_P)/2)$.
P'_C has coordinates
$((x_C + x_P)/2, (y_C - y_P)/2)$.
The circumcircle of ABC is given by the equation
$(x-1/2)^2+(y-(y_C^2+(x_C-1)*x_C)/(2*y_C))^2-1/4-((y_C^2+(x_C-1)*x_C)/(2*y_C))^2 = 0$.
The nine point circle of ABC is given by the equation
$(x - (x_C/2+1/4))^2 + (y - ((-x_C^2 + x_C + y_C^2)/(4 * y_C)))^2 - (1/2 - (x_C/2+1/4))^2 - (((-x_C^2 + x_C + y_C^2)/(4 * y_C)))^2 = 0$.
If P lies on the circumcircle, we can express its coordinates as either
$(x_P, (x_C^2 - x_C + y_C^2 + (-4 * x_P^2 * y_C^2 + 4 * x_P * y_C^2 + x_C^4 - 2 * x_C^3 + 2 * x_C^2 * y_C^2 + x_C^2 - 2 * x_C * y_C^2 + y_C^4)^{1/2})/(2 * y_C))$
or
$(x_P, -(x_C - x_C^2 - y_C^2 + (-4 * x_P^2 * y_C^2 + 4 * x_P * y_C^2 + x_C^4 - 2 * x_C^3 + 2 * x_C^2 * y_C^2 + x_C^2 - 2 * x_C * y_C^2 + y_C^4)^{1/2})/(2 * y_C))$.
In the first case, we verify that P'_A, P'_B, and P'_C lie on the nine point circle since we calculate
$((2*(y_C*((x_C-1)*((x_C^2-x_C+y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C))+y_C)+(x_C-1)^2*x_P)/(y_C^2+(x_C-1)^2)-x_P)/2-(x_C/2+1/4))^2+((2*(y_C*(y_C*((x_C^2-x_C+y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C))+(x_C-1)*x_P-x_C+1))/(y_C^2+(x_C-1)^2)-((x_C^2-x_C+y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C)))/2-((-x_C^2+x_C+y_C^2)/(4*y_C)))^2-(1/2-(x_C/2+1/4))^2-(((-x_C^2+x_C+y_C^2)/(4*y_C)))^2=0$,
$((1+2*(x_C*(x_C*x_P+y_C*((x_C^2-x_C+y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C))))/(x_C^2+y_C^2)-x_P)/2-(x_C/2+1/4))^2+((2*(y_C*(x_C*x_P+y_C*((x_C^2-x_C+y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C))))/(x_C^2+y_C^2)-((x_C^2-x_C+y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C)))/2-((-x_C^2+x_C+y_C^2)/(4*y_C)))^2-(1/2-(x_C/2+1/4))^2-(((-x_C^2+x_C+y_C^2)/(4*y_C)))^2=0$,
$((x_C+x_P)/2-(x_C/2+1/4))^2+((y_C-((x_C^2-x_C+y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C)))/2-((-x_C^2+x_C+y_C^2)/(4*y_C)))^2-(1/2-(x_C/2+1/4))^2-(((-x_C^2+x_C+y_C^2)/(4*y_C)))^2=0$,
and in the second case we verify that P'_A, P'_B, and P'_C lie on the nine point circle since we calculate
$((2*(y_C*((x_C-1)*(-(x_C-x_C^2-y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C))+y_C)+(x_C-1)^2*x_P)/(y_C^2+(x_C-1)^2)-x_P)/2-(x_C/2+1/4))^2+((2*(y_C*(y_C*(-(x_C-x_C^2-y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C))+(x_C-1)*x_P-x_C+1))/(y_C^2+(x_C-1)^2)-(-(x_C-x_C^2-y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C)))/2-((-x_C^2+x_C+y_C^2)/(4*y_C)))^2-(1/2-(x_C/2+1/4))^2-(((-x_C^2+x_C+y_C^2)/(4*y_C)))^2=0$,
$((1+2*(x_C*(x_C*x_P+y_C*(-(x_C-x_C^2-y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+$

$2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C))))/(x_C^2+y_C^2)-x_P)/2-(x_C/2+1/4))^2+((2*(y_C*(x_C*x_P+y_C*(-(x_C-x_C^2-y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C))))/(x_C^2+y_C^2)-(-(x_C-x_C^2-y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C)))/2-((-x_C^2+x_C+y_C^2)/(4*y_C)))^2-(1/2-(x_C/2+1/4))^2-(((-x_C^2+x_C+y_C^2)/(4*y_C)))^2=0,$
$((x_C+x_P)/2-(x_C/2+1/4))^2+((y_C-(-(x_C-x_C^2-y_C^2+(-4*x_P^2*y_C^2+4*x_P*y_C^2+x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4)^{1/2})/(2*y_C)))/2-((-x_C^2+x_C+y_C^2)/(4*y_C)))^2-(1/2-(x_C/2+1/4))^2-(((-x_C^2+x_C+y_C^2)/(4*y_C)))^2=0$
as desired. □

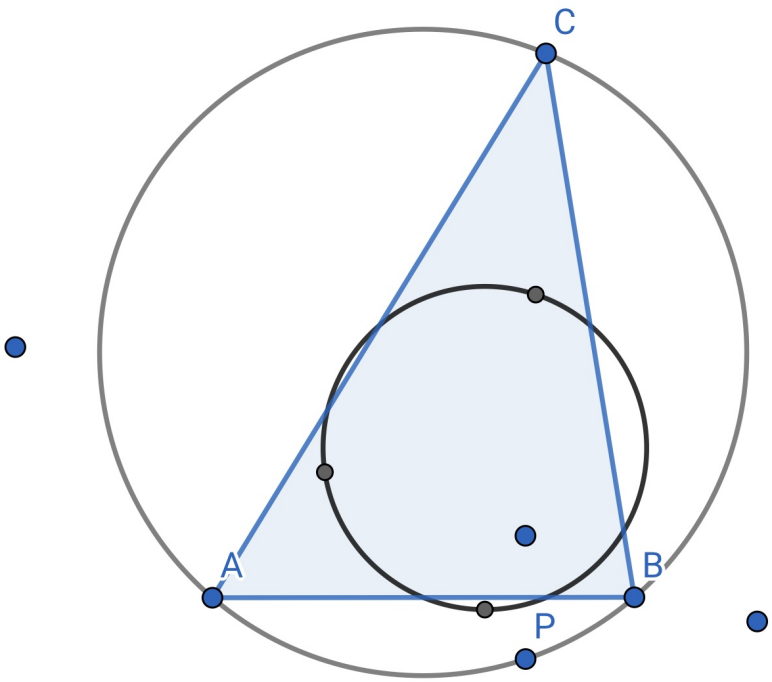

Figure 1. Illustration.

References

[1] Weisstein, Eric W. "Nine-Point Circle." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Nine-PointCircle.html