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Computer Discovered Mathematics: Gibert Triangles

Sava Grozdev[a] and Deko Dekov[b2]
[a] VUZF University of Finance, Business and Entrepreneurship,
Gusla Street 1, 1618 Sofia, Bulgaria
e-mail: sava.grozdev@gmail.com
[b]Zahari Knjazheski 81, 6000 Stara Zagora, Bulgaria
e-mail: ddekov@ddekov.eu
web: http://www.ddekov.eu/

Abstract. The following result is well known (See Gibert [4, Glossary, Brocard Triangles]): The vertices of the Second Brocard Triangle are the projections of the Circumcenter on the symmedians of the Reference triangle. In this paper we study the following generalization: Given points P and U. We call *Gibert triangle of points P and U* the triangle whose vertices are the projections of point P on the cevians of point U. Now is the time of computers. We use the computer program "Discoverer". The results discovered by the "Discoverer", and presented here as the Supplementary Material, give us the possibility we to find 1109 different ways for constructing the Second Brocard Triangle by using compass and ruler, and 875 different ways for constructing the Fourth Brocard Triangle.

Keywords. Gibert triangle, triangle geometry, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See [5].

Given triangle ABC. The following result is well known (See Gibert [4, Glossary, Brocard Triangles]): The vertices of the Second Brocard Triangle are the projections of the Circumcenter on the symmedians.

[2]Corresponding author

In this paper we study the following generalization: Given points P and U. We call *Gibert triangle of points P and U* the triangle whose vertices are the projections of point P on the cevians of point U.

The theorems in this paper are discovered by the computer program "Discoverer".

Figure 1 illustrates the definition of the Gibert triangle. In Figure 1, P and U are arbitrary points, Ta, Tb and Tc are the projections of P on the lines AU, BU and CU, respectively. Then triangle $TaTbTc$ is the Gibert triangle of P and U.

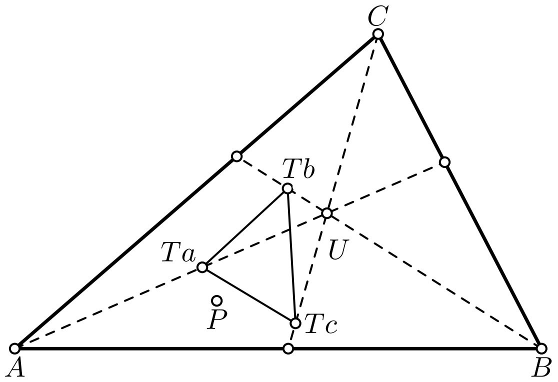

FIGURE 1.

2. PRELIMINARIES

In this section we review some basic facts about barycentric coordinates. We refer the reader to [15],[3],[2],[11],[7],[8],[10],[13]. The labeling of triangle centers follows Kimberling's ETC [9]. For example, X(1) denotes the Incenter, X(2) denotes the Centroid, X(37) is the Grinberg Point, X(75) is the Moses point (Note that in the prototype of the "Discoverer" the Moses point is the X(35)), etc.

The reader may find definitions in [6, Contents, Definitions], and in [14].

The reference triangle ABC has vertices $A = (1, 0, 0), B(0, 1, 0)$ and C(0,0,1). The side lengths of $\triangle ABC$ are denoted by $a = BC,\ b = CA$ and $c = AB$. We use the Conway's notation:

$$S_A = \frac{b^2 + c^2 - a^2}{2},\ S_B = \frac{c^2 + a^2 - b^2}{2},\ S_C = \frac{a^2 + b^2 - c^2}{2},$$

Three points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$ lie on the same line if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0. \tag{1}$$

The equation of the line joining two points with coordinates u_1, v_1, w_1 and u_2, v_2, w_2 is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0. \tag{2}$$

The intersection of two lines $L_1 :\ p_1x + q_1y + r_1z = 0$ and $L_2 :\ p_2x + q_2y + r_2z = 0$ is the point

$$(q_1r_2 - q_2r_1, r_1p_2 - r_2p_1, p_1q_2 - p_2q_1) \tag{3}$$

The equation of the line through point $P(u,v,w)$ and perpendicular to the line $L: px+qy+rz=0$ is as follows (The method discovered by Floor van Lamoen):

$$\begin{vmatrix} F & G & H \\ u & v & w \\ x & y & z \end{vmatrix} = 0 \tag{4}$$

where $F = S_B g - S_C h$, $G = S_C h - S_A f$, and $H = S_A f - S_B g$.

Given a point $P(u,v,w)$, the complement of P is the point $(v+w, w+u, u+v)$, the anticomplement of P is the point $(-u+v+w, -v+w+u, -w+u+v)$, the isotomic conjugate of P is the point (vw, wu, uv), and the isogonal conjugate of P is the point (a^2vw, b^2wu, c^2uv).

3. Gibert Triangles

Theorem 3.1. *Let P = (p,q,r) and U = (u,v,w) be points in barycentric coordinates. Then the barycentric coordinates of the Gibert triangle $TaTbTc$ of points P and U are as follows:*

$$uTa = wb^2vq - wqc^2v - wa^2vq - qw^2c^2 + qw^2a^2 + qw^2b^2 + 2pw^2b^2 - 2wpa^2v$$
$$+2wpb^2v + 2wpc^2v - rb^2v^2 + rc^2v^2 + ra^2v^2 + vwc^2r - vwa^2r - vrwb^2 + 2pc^2v^2,$$
$$vTa = -v(-2rwb^2 + ra^2v - rb^2v - rc^2v + qwa^2 - qwb^2 - qwc^2 - 2qc^2v),$$
$$wTa = -w(-2rwb^2 + ra^2v - rb^2v - rc^2v + qwa^2 - qwb^2 - qwc^2 - 2qc^2v).$$
$$uTb = u(2wa^2r + pwa^2 - pwb^2 + pwc^2 - rub^2 + ruc^2 + rua^2 + 2puc^2),$$
$$vTb = uwc^2r - uwa^2r - urwb^2 - ru^2a^2 + ru^2b^2 + ru^2c^2 + 2qu^2c^2 - 2uqwb^2$$
$$+2uqwc^2 + 2uqwa^2 + 2qw^2a^2 + pw^2b^2 - pw^2c^2 + pw^2a^2 + wpua^2 - wpub^2 - wpuc^2,$$
$$wTb = w(2wa^2r + pwa^2 - pwb^2 + pwc^2 - rub^2 + ruc^2 + rua^2 + 2puc^2).$$
$$uTc = u(-quc^2 + ua^2q + 2pub^2 + ub^2q + 2a^2vq + pa^2v + pb^2v - pc^2v),$$
$$vTc = v(-quc^2 + ua^2q + 2pub^2 + ub^2q + 2a^2vq + pa^2v + pb^2v - pc^2v),$$
$$wTc = 2vrub^2 - 2vuc^2 + 2vrua^2 - vpuc^2 + vpua^2 - vpub^2 - pb^2v^2 + pa^2v^2$$
$$+2ra^2v^2 + pc^2v^2 + 2ru^2b^2 - u^2a^2q + u^2b^2q + qu^2c^2 + ub^2vq - uqc^2v - ua^2vq.$$

Proof. Denote $P = (p,q,r)$ and $U = (u,v,w)$. By using (2) we find the barycentric equation of the line L_1 through A and U. By using (4) we find the barycentric equation of the line L_2 through point P and perpendicular to line L_1. Then by using (3) we find the point of intersection of lines L_1 and L_2. The last point is the vertex Ta of the Gibert triangle $TaTbTc$ of P and Q. Similarly, we find the vertices Tb and Tc. The barycentric coordinates of points Ta, Tb and Tc are given in the statement of the theorem. □

Let $KaKbKc$ is the circumcevian triangle of an arbitrary point U. Then the *Half-Circumcevian Triangle of U* is the triangle whose vertices are the midpoints of segments AKa, BKb and CKc, respectively.

Theorem 3.2. *The Gibert Triangle of the Circumcenter and an arbitrary point U coincides with the Half-Circumcevian Triangle of point U.*

Figure 2 illustrates Theorem 3.2. In Figure 2, O is the Circumcenter, U is an arbitrary point, $TaTbTc$ is the Gibert triangle of the Circumcenter and point U, and $KaKbKc$ is the Circumcevian triangle of point U. Then triangle $TaTbTc$ is the Half-Circumcevian triangle of point U.

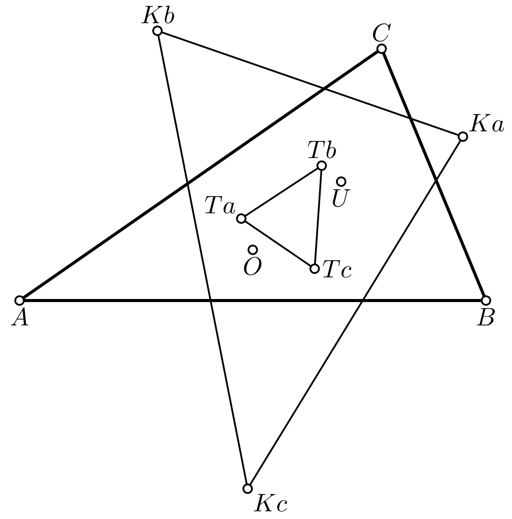

FIGURE 2.

4. Special Case: Second Brocard Triangle

We obtain as a special case of Theorem 3.1:

Theorem 4.1. *The Gibert Triangle of the Circumcenter and the Symmedian Point is the Second Brocard Triangle.*

Folder 1 of the enclosed Supplementary Material contains triangles perspective with the Second Brocard Triangle.

5. Special Case: Fourth Brocard Triangle

Recall that the Fourth Brocard triangle is the isogonal conjugate of the Second Brocard Triangle (see [4, Glossary, Brocard Triangles]). Another names of the Fourth Brocard Triangle are as follows: D-triangle in [14], Johnson Triangle in [1] and Involutory Triangle in [7].

As a special case of Theorem 3.1 we obtain (See also [4, Glossary, Brocard Triangles]):

Theorem 5.1. *The Gibert Triangle of the Orthocenter and the Centroid is the Fourth Brocard Triangle.*

Folder 2 of the enclosed Supplementary Material contains triangles perspective with the Fourth Brocard Triangle. Table 1 lists a few of the perspectors.

X(n)	**Perspector**	**The Fourth Brocard Triangle is perspective with the**
X(2)	Centroid	Triangle ABC
X(15)	First Isodynamic Point	Outer Napoleon Triangle
X(16)	Second Isodynamic Point	Inner Napoleon Triangle
X(8877)	X(8877)	Anticevian Triangle of the Parry Point

Table 1

The theorems in Folder 2, Table X-P, could be proved by using the following sample:

Theorem 5.2 (Folder 2, Table X-P, row 1)**.** *Triangle ABC and the Fourth Brocard Triangle are perspective and the perspector is the Centroid.*

Proof. The barycentric coordinates of the Fourth Brocard Triangle $TaTbTc$ are as follows:

$$Ta = (a^2, -a^2 + b^2 + c^2, -a^2 + b^2 + c^2),\ Tb = (-b^2 + c^2 + a^2, b^2, -b^2 + c^2 + a^2),$$
$$Tc = (-c^2 + a^2 + b^2, -c^2 + a^2 + b^2, c^2).$$

The Centroid has barycentric coordinates $G = (1, 1, 1)$. By using (1) we prove that points A, G and Ta lie on the same line. We obtain

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ a^2 & -a^2 + b^2 + c^2 & -a^2 + b^2 + c^2 \end{vmatrix} = 0.$$

Similarly, we prove that points B, G and Tb lie on the same line, and the points C, G and Tc lie on the same line. Hence, point G is the perspector of triangles ABC and $TaTbTc$. □

6. Geometric Constructions

Geometric constructions by compass and ruler are an important part of the high-school education.

Suppose that we want to construct a triangle $TaTbTc$ by using compass and ruler. In previous papers we have described the following two methods:

Method 1.
Suppose that we can construct a triangle $KaKbKc$ which is perspective with triangle $TaTbTc$ with perspector K, and a second triangle $MaMbMc$ perspective with triangle $TaTbTc$ with perspector M ($K \neq M$). Then we can construct the triangle $TaTbTc$ as follows: The vertex Ta is the intersection of the lines KKa and MMa. Similarly, vertex Tb is the intersection of the lines KKb and MMb, and vertex Tc is the intersection of the lines KKc and MMc.

Method 2.
Suppose that we can construct the circumcircle of triangle $TaTbTc$ and a triangle $KaKbKc$ which is perspective with triangle $TaTbTc$ with perspector K. Then we can construct triangle $TaTbTc$ as follows: The vertex Ta is the intersection of the circumcircle of triangle $TaTbTc$ and the line KKa. Similarly, vertex Tb is the intersection of the circumcircle of triangle $TaTbTc$ and the line KKb, and vertex Tc is the intersection of the circumcircle of triangle $TaTbTc$ and the line KKc.

The above methods are especially effective with the computer program "Discoverer".

For example, if we use Method 1, the enclosed results give us the possibility to construct 1040 different ways how to construct the Second Brocard Triangle, and 809 different way how we to construct the Fourth Brocard Triangle.

The circumcircle of the Second Brocard Triangle is the Brocard Circle, and the circumcircle of the Fourth Brocard Triangle is the Orthocentroidal circle. By using these results, and by using the enclosed results, we see that the Method 2 give us the possibility to find 69 different ways how to construct the Second Brocard Triangle, and 66 different way how to construct the Fourth Brocard Triangle.

Example 6.1. *The Fourth Brocard Triangle $TaTbTc$ and triangle ABC are perspective with perspector the Centroid (Folder 2, Table X-P, row 1). Also, the Fourth Brocard Triangle and the Outer Napoleon Triangle are perspective with perspector the First Isodynamic Point (Folder 2, Table X-P, row 57). Hence, we can construct the Fourth Brocard Triangle by using Method 1.*

Problem for the reader:

Problem 6.1. *Describe all 1040 different ways for constructing the Second Brocard Triangle by using Method 1. Describe all 809 different ways for constructing the Fourth Brocard Triangle by using Method 1.*

Example 6.2. *The Fourth Brocard Triangle $TaTbTc$ and triangle ABC are perspective with perspector the Centroid (Folder 2, Table X-P, row 1). Also, the Orthocentroidak circle is the circumcircle of the Fourth Brocard Triangle. Hence, we can construct the Fourth Brocard Triangle by using Method 2.*

Supplementary Material

The enclosed supplementary material contains theorems related to the topic.

Acknowledgement

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