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Problem of Twelve Circles

Nguyen Trung Kien[a2] and Tran Thu Le[b]
[a] Mathematics Department, Teacher College,
Cantho University, Vietnam.
e-mail: trungkien@ctu.edu.vn
[b] Student at Mathematics Department, Teacher College,
Cantho University, Vietnam.
e-mail: letranthu23041996@gmail.com

Abstract. In this note we aim to propose and solve the problem of twelve circles. It shows a nice property of cycles which appear in the so-called Apollonius' problem.

Keywords. Apollonius' problem; Homothetic transformation; Inverse operation.

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

What we refer to as the problem of Apollonius today is one subject of the two lost books, namely The Tangencies, by Apollonius of Perga (ca. 262 – 190 BC). In the 4th century Pappus of Alexandria, a famous geometer, published the multivolumn to make the work of Apollonius survived. From this time on, the problem of Apollonius has been intensively studiedand generalized; see [3, 4, 5]. In the Apollonius' problem, a nice property can be stated as follows; see [1] and Fig. 1.

'Three given circles generically have eight different circles that are tangent to them and each solution circle encloses or excludes the three given circles in a different way: in each solution, a different subset of the three circles is enclosed (its complement is excluded) and there are 8 subsets of a set whose cardinality is 3, since $8 = 2^3$.'

Note that the Apollonius' system includes three given circles generically and eight different circles are tangent to them. In this note we construct and shows a nice property in which twelve different circles are tangent to the three given circles.

[2]Corresponding author

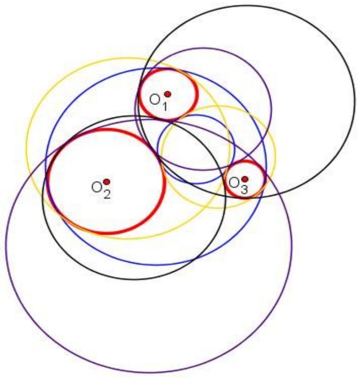

FIGURE 1. $(O_1), (O_2), (O_3)$ are the three given circles and eight different circles that are tangent to them.

2. THE PROBLEM OF TWELVE CIRCLES

For simplicity, we introduce some basic definitions. We shortly name a circle as (O) whose center is at O. We say (I), (J) are conjugate circles on (O) if (O) is internal tangent to (O) and (J) is external tangent to (O). Generally speaking, two circles (I), (J) are conjugate circles on (O_1), (O_2) ,..., (O_k) if (I) and (J) are conjugate on each circle (O_i) for $i = 1, \dots, k$. Therefore, we can devide 8 circles in the Apollonius problem to 4 conjugate pairs circles. Let us now define the theorem of twelve circles.

Theorem of twelve circles: Given an Apollonius' system on the plane with 3 circles (O_1), (O_2), (O_3) and 4 pairs of arbitrary circles. Then lines passing through 2 centers of pairs of circles are concurrent at the radical center of (O_1), (O_2), (O_3), say K. Moreover, the intersection of each pair of circles (if there exists) lies on a circle centered at K.
Proof.

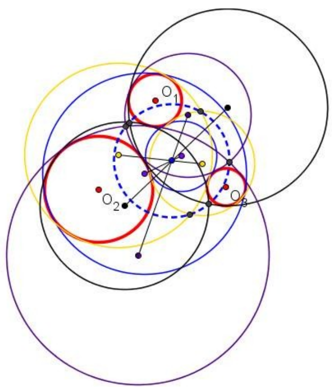

FIGURE 2. The problem of twelve circles.

To prove this main theorem, we propose the following claim.

Claim: Let (I), (J) be a conjugate pair of cirlces on (O_1), (O_2). Denote by P and Q the intersections of (I) and (J); K the internal homothetic center of (I) and (J). Then K lies on the radical axis of (I) and J. Moreover, $KP = KQ = \sqrt{P_K(O)}$.
Proof of the claim.
Let M, M' be the common points of (O_1) and (I), (J), respectively. Furthermore,

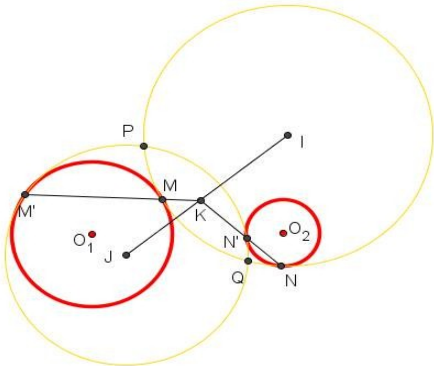

Figure 3. An instance of (I) and (J).

let N, N' be the common points of (O_2) and (I), (J). As (I) and (J) are conjugate circles on (O_1), (O_2) , we assume without loss of generality that (I) is external tangent to (O_1) and (J) is internal tangent to (O_2). The following holds:

- M is the internal homothetic center of (O_1) and I,
- M' is the external homothetic center of (O_1) and J,
- K is the internal homothetic center of (I) and J.

Thus, three points M, M', K are collinear according to Monge-D'Alembert 2 Theorem (see [2]). By the same argument, N, N', K are also collinear. Therefore, the two lines MM', NN' pass through K.

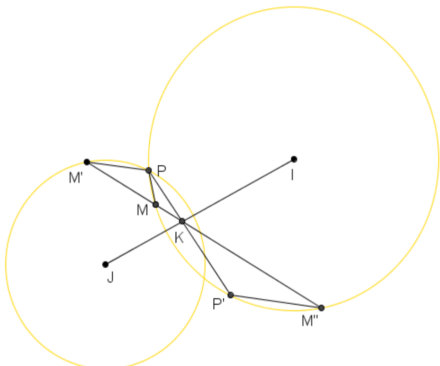

Figure 4. An instance of the claim.

Let KP and MM' intersect (I) at P' and M", respectively. As (I) and (J) are conjugate circles on (O_1), (O_2), we obtain the two collinear triples (O_1, I, M), (O_1, J, M'). Hence, O_1 is the intersection of IM and JM'. Consider a map f which is a homothetic transformation centered at K mapping (I) to (J). As IM can not be parallel to JM', we obviously get $M' = f(M'')$ and $P = f(P')$. As $PM' \parallel P'M''$, one has

$$\frac{\overline{KP}}{\overline{KP'}} = \frac{\overline{KM'}}{\overline{KM''}} \tag{1}$$

As P, P', M, M'' lies on (I), it holds

$$\overline{KP}\,\overline{KP'} = \overline{KM}\,\overline{KM''} \tag{2}$$

By (1) and (2), we obtain the equality $KP^2 = \overline{KM}\,\overline{KM'}$. It yields $KQ^2 = \overline{KN}\,\overline{KN'}$ by the similar argument. Finally, we get a sequence of equalities as follows.

$$KP^2 = KQ^2 = \overline{KM}\,\overline{KM'} = \overline{KN}\,\overline{KN'} = P_K(O_1) = P_K(O_2).$$

Insummary, K is contained in radical axis of (O_1), (O_2) and $KP = KQ = \sqrt{P_K(O)}$. The claim has been proved.

We are back to the theorem of twelve cycles. Consider (I) and (J) as a pair of circles in the Apollonius' system. Denote by K is the internal homothetic center of (I) and (J). By the previous claim, K is contained in the radical axis of (O_1) and (O_2) . Similarly, K is also contained in the radical axis of (O_2) and (O_3); (O_1) and (O_3). We can also say that, K is the radical center of (O_1), (O_2), (O_3). Therefore, the lines passing through two centers of each pair of circles are concurrent at K. Moreover, the intersections of pairs of circles are contained in the circle centered at K and radius $\sqrt{P_K(O_1)}$. This twelfth circle in Apollonius' system is the radical circle (see [3]) of (O_1), (O_2), (O_3).

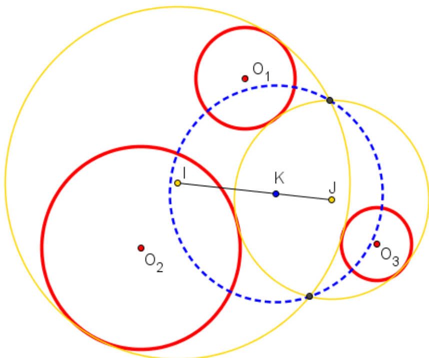

FIGURE 5. K is contained in the radical axis of (O_2) and (O_3).

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