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Triangles Homothetic with Triangle ABC. Part 3

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Abstract. By using the computer program "Discoverer" we study triangles homothetic with triangle ABC. This is the third part of the investigation.

Keywords. triangle geometry, remarkable point, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

We continue the study of triangles homothetic with triangle ABC. For the first two part of this investigation see [8], [7].

Given a point P in the plane of triangle ABC. Denote by $TaTbTc$ the Triangle of Reflections of the vertices of the Medial triangle of triangle ABC in point P. Then triangles ABC and $TaTbTc$ are homothetic.

Figure 1 illustrates the above construction. In figure 1,

- $MaMbMc$ is the Medial triangle of triangle ABC,
- Ta is the reflection of point Ma in point P,
- Tb is the reflection of point Mb in point P,
- Tc is the reflection of point Mc in point P,

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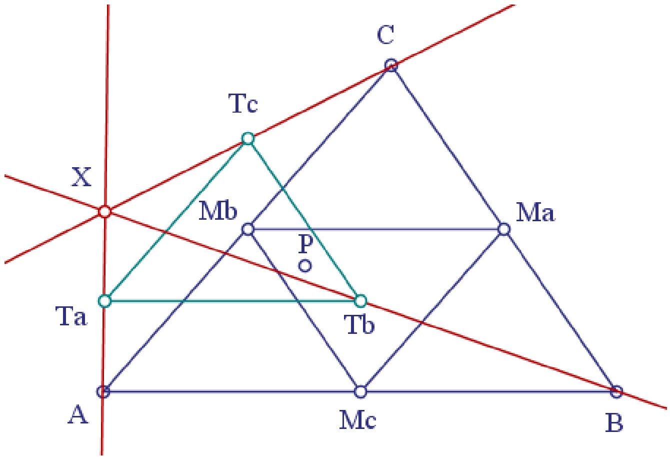

FIGURE 1.

- $TaTbTc$ is the Triangle of Reflections of the vertices of triangle $MaMbMc$ in point P,
- X is the center of homothety of triangles ABC and $TaTbTc$.

We use barycentric coordinates. See [1] - [18]. We use the computer program "Discoverer" created by the authors.

2. GENERAL CASE

Let $P = (u, v, w)$. Denote by $T(P) = TaTbTc$ the Triangle of Reflections of the vertices of the Medial triangle of triangle ABC in point P

Theorem 2.1. *The barycentric coordinates of triangle $T(P)$ are as follows:*

$$\begin{aligned} Ta &= (4u, 3v - u - w, 3w - u - v), \\ Tb &= (3u - v - w, 4v, 3w - u - v), \\ Tc &= (3u - v - w, 3v - u - w, 4w). \end{aligned}$$

Proof. We use the barycentric coordinates of the Medial triangle and the reflection formula (15) in [5]. □

Theorem 2.2. *Triangle $T(P)$ is homothetic with triangle ABC. The center of homothety X has barycentric coordinates*

$$X = (3u - v - w, 3v - u - w, 3w - u - v).$$

The ratio of homothety is $\frac{1}{2}$.

Proof. We use the homothety formula (17) in [5]. □

Theorem 2.3. *The center of homothety of triangles ABC and $T(P)$ is the anticomplement of anticomplement of point P.*

Proof. We use Theorem 2.2 and the formula for the anticomplement in barycentric coordinates. □

Note that the above theorem gives simple construction of the center of homothety of triangles ABC and $T(P)$ as well as simple calculation of its barycentric coordinates.

Theorem 2.4. *Given points $P = (u, v, w)$ and $Q = (p, q, r)$. Then the homothetic image $hQ = (p_1, q_1, r_1)$ of Q under the homothety of triangles ABC and $T(P)$ is*

$$\begin{aligned} p_1 &= 4up + 3uq + 3ur - vq - vr - wq - wr, \\ q_1 &= 3vp + 4vq + 3vr - up - ur - wp - wr, \\ r_1 &= 3wp + 3wq + 4wr - up - uq - vp - vq. \end{aligned}$$

Proof. We use the homothety formula (17) in [5]. □

By using the computer program "Discoverer", we have investigated 102 centers of homothety. Of these 23 centers of homothety are Kimberling points and the rest of 79 centers are new points. See the Complementary material.

Table 1 lists centers of homothety of triangles ABC and $T(P)$ in terms of Kimberling points $X(n)$.

	Point P	**Center of homothety of triangles ABC and $T(P)$**
1	Incenter	X(145)
2	Centroid	X(2)
3	Circumcenter	X(20)
4	Orthocenter	X(3146)
5	Symmedian Point	X(193)
6	Nagel Point	X(3621)
7	Mittenpunkt	X(144)
8	Spieker Center	X(8)
9	Feuerbach Point	X(149)
10	de Longchamps Point	X(5059)
11	Grinberg Point	X(192)
12	Brocard Midpoint	X(194)
13	Tarry Point	X(5984)
14	Kiepert Center	X(148)
15	Second Mid-Arc Point	X(7057)
16	Center of the Brocard Circle	X(6776)
17	Equal Parallelians Point	X(4788)
18	Weill Point	X(4430)
19	Center of the Orthocentroidal Circle	X(3543)
20	Conway Point	X(6658)
21	Center of the Taylor Circle	X(5889)
22	Outer Vecten Point	X(12222)
23	Inner Vecten Point	X(12221)

Table 1.

Table 1 is an addition to results in Kimberling [11].

Also, we can investigate the new points, that is, the points which are not available in Kimberling. For example, denote by $D = (uD, vD, wD)$ the center of homothety of triangles ABC and $T(X(7))$. Then

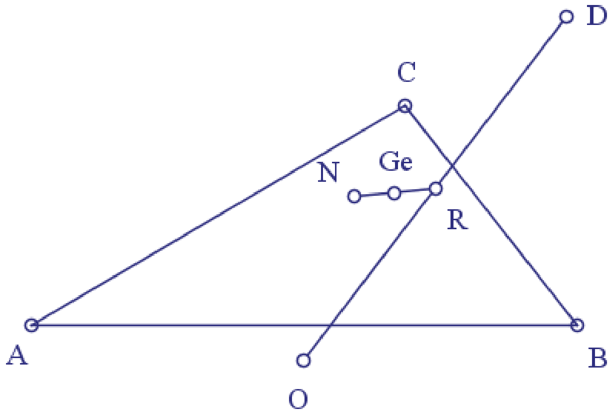

FIGURE 2.

Theorem 2.5. *The barycentric coordinates of point D are as follows:*

$$\begin{aligned} uD &= 5a^2 - 3b^2 + 6bc - 3c^2 - 2ac - 2ab, \\ vD &= 6ac - 3a^2 + 5b^2 - 3c^2 - 2bc - 2ab, \\ wD &= 6ab - 3b^2 + 5c^2 - 3a^2 - 2bc - 2ac. \end{aligned}$$

Theorem 2.6. *Point D is the*

(1) *Reflection of the Circumcenter in the Reflection of the Nine-Point Center in the Gergonne Point.*
(2) *Reflection of the Orthocenter in the Reflection of the Circumcenter in the Gergonne Point.*
(3) *Reflection of the Nagel Point in the Reflection of the Incenter in the Gergonne Point.*
(4) *Reflection of the de Longchamps Point in the Reflection of the Orthocenter in the Gergonne Point.*
(5) *Reflection of the Retrocenter in the Reflection of the Symmedian Point in the Gergonne Point.*

Figure 2 illustrates part (1) of Theorem 2.6. In figure 2,

- N is the Nine-Point Center,
- Ge is the Gergonne point,
- R is the reflection of N in Ge,
- O is the Circumcenter,
- D is the reflection of O in R.

Theorem 2.7. *Point D lies on the following lines:*

(1) *Line through the Centroid and the Gergonne Point.*
(2) *Line through the Circumcenter and the Reflection of the Nine-Point Center in the Gergonne Point.*
(3) *Line through the Orthocenter and the Reflection of the Circumcenter in the Gergonne Point.*
(4) *Line through the Nagel Point and the Reflection of the Incenter in the Gergonne Point.*
(5) *Line through the de Longchamps Point and the Reflection of the Orthocenter in the Gergonne Point.*

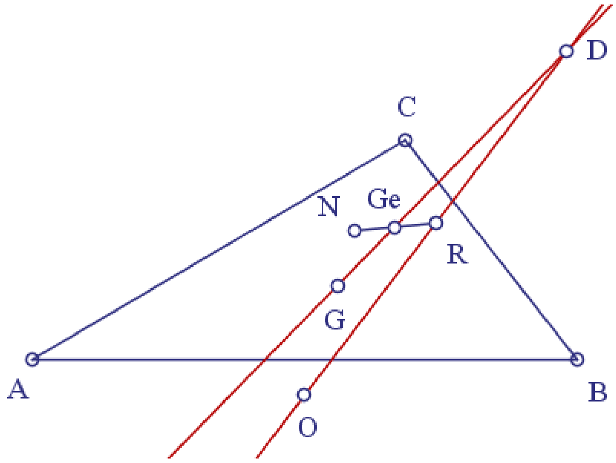

FIGURE 3.

(6) *Line through the Retrocenter and the Reflection of the Symmedian Point in the Gergonne Point.*
(7) *Line through the Reflection of the Nagel Point in the Gergonne Point and the Reflection of the Nagel Point in the Incenter.*

Figure 3 illustrates the first part of Theorem 2.7. In figure 3, point D lines on the line GGe where G is the Centroid and Ge is the Gergonne point.

Note the above two theorems, as well as Theorem 2.3 give simple constructions of point D.

Problem 1. *Find the barycentric coordinates and the propertes of new points available in the Supplementary material.*

3. SPECIAL CASE P = INCENTER

3.1. **Triangle T(X(1)).** We study the special case when point P is the Incenter = X(1). We recommend the reader to study other special cases, e.g. when P is the Centroid, or Circumcenter, or Orthocenter.

We denote by T(X(1)) the triangle T(P) for the special case when P is the Incenter.

The Incenter has barycentric coordinates $I = (a, b, c)$, so that from Theorem 2.1 we obtain immediately the barycentric coordinates of triangle T(X(1)).

About Euler triangle of point P see [6].

Theorem 3.1. *Triangle T(X(1)) is the Euler triangle of the Reflection of the Nagel Point in the Incenter.*

Figure 4 illustrates Theorem 3.1. In figure 4,

- N is the Nagel point,
- I is the Incenter,
- R is the reflection of point N in point I.
- $EaEbEc$ is the Euler triangle of point R.

Triangles $EaEbEc$ and $T(X(1))$ coincide.

Proof. Direct calculation. □

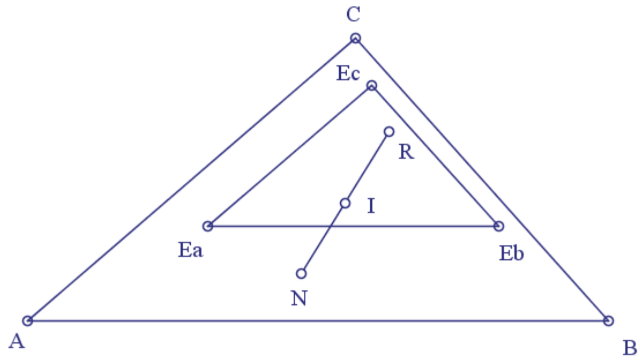

FIGURE 4.

3.2. **Perspectors.** We have investigated 159 triangles perspective with triangle T(X(1)). We have found 25 perspectors which are Kimberling points. The rest of 124 perspectors are new points. See the Complementary material. Below we list three of the perspectors which are Kimberling points.

Theorem 3.2. *The perspector of triangle T(X(1)) and the*

(1) *Anticevian Triangle of the Mittenpunkt is the X(1).*
(2) *Pedal Triangle of the Reflection of the Circumcenter in the Incenter is the X(1482).*
(3) *Cevian Triangle of the Reflection of the Nagel Point in the Incenter is the X(145).*

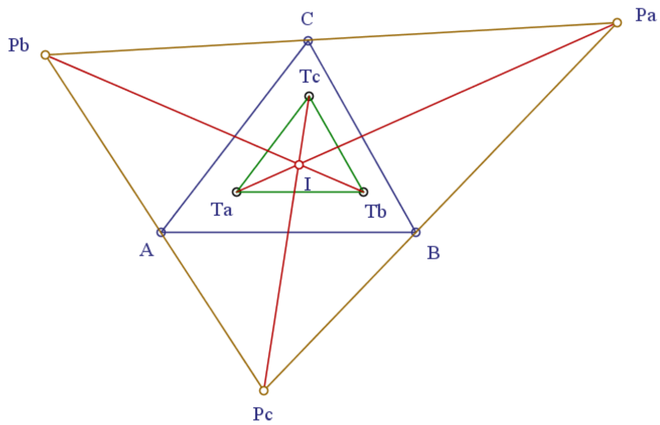

FIGURE 5.

Figure 5 illustrates Theorem 3.2. In figure 5,

- $TaTbTc$ is triangle T(X(1)),
- $PaPbPc$ is the Anticevian Triangle of the Mittenpunkt,
- I is the Incenter.

The Incenter is the perspector of triangles T(X(1)) and $PaPbPc$.

We encourage the reader to investigate the perspectors which are new points, that is, points not available in Kimberling [11]. A list of these new points is available in the Supplementary material.

3.3. **Kimberling Points.** We have investigated 195 notable points of triangle T(X(1)). Of these 13 are Kimberling points and the rest of 182 points are new points. See the Complementary material.

Table 2 lists a few notable points of triangle T(X(1)) as Kimberling points $X(n)$.

	Point of Triangle T(X(1))	**Point of Triangle ABC**
X(1)	Incenter	X(3244)
X(2)	Centroid	X(3241)
X(3)	Circumcenter	X(1483)
X(4)	Orthocenter	X(1482)
X(7)	Gergonne Point	X(3243)
X(8)	Nagel Point	X(1)
X(10)	Spieker Center	X(3635)
X(20)	de Longchamps Point	X(944)
X(40)	Bevan Point	X(5882)
X(69)	Retrocenter	X(3242)
X(72)	Quotient of the Grinberg Point and the Orthocenter	X(9957)
X(100)	Anticompliment of the Feuerbach Point	X(1317)
X(355)	Center of the Fuhrmann Circle	X(10222)

Table 2.

Table 2 is an addition to results in Kimberling [11].

We encourage the reader investigate the new points, that is, the points which are not available in Kimberling [11].

3.4. **Circles.** We have investigated 29 notable circles of triangle T(X(1)). See the Complementary material.

Table 3 lists a few notable circles of Triangle T(X(1)) whose centers are Kimberling points $X(n)$.

	Circle C	**Center of Circle C as Kimberling point**
1	Circumcircle of Triangle T(X(1))	X(1483)
2	Incircle of Triangle T(X(1))	X(3244)
3	Excentral Circle of Triangle T(X(1))	X(5882)
4	Antimedial Circle of Triangle T(X(1))	X(1482)
5	Spieker Circle of Triangle T(X(1))	X(3635)
6	Second Brocard Circle of Triangle T(X(1))	X(1483)

Table 3.

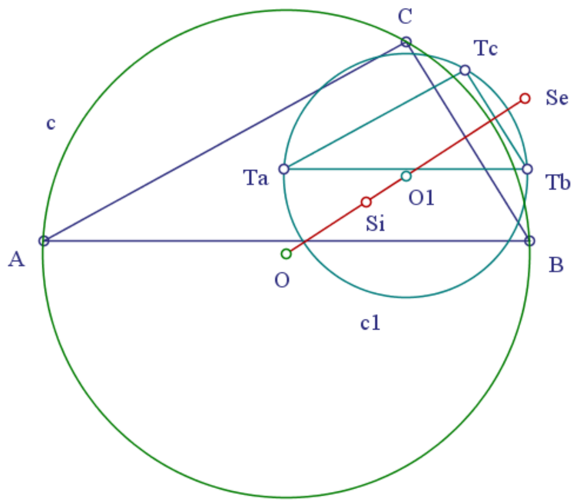

FIGURE 6.

3.5. **Similitude Centers.** We have investigated 836 External Similitude Centers. Of these 40 are Kimberling points and the rest of 796 are new points. See the Supplementary material. Below are two examples:

Theorem 3.3. *The External Center of Similitude of the Circumcircle of Triangle ABC and the Circumcircle of Triangle T(X(1)) is the point X(145), that is, the Center of homothety of triangles ABC and T(X(1)).*

We have investigated 841 Internal Similitude Centers. Of these 20 are Kimberling points and the rest of 821 are new points. See the Supplementary material.

Theorem 3.4. *The Internal Center of Similitude of the Circumcircle of Triangle ABC and the Circumcircle of Triangle T(X(1)) is the point X(7967). The first barycentric coordinate of point X(7967) is as follows:*

$$4a^2b^2+4a^2c^2+4a^3b-4ab^3+4a^3c-2b^2c^2-4ac^3+4abc^2-8a^2bc+4ab^2c-5a^4+b^4+c^4$$

Figure 6 illustrates Theorems 3.3 and 3.4. In figure 6,

- c is the circumcircle of triangle ABC,
- O is the circumcenter of triangle ABC,
- $TaTbTc$ is triangle T(X(1)),
- $c1$ is the circumcircle of triangle $TaTbTc$,
- $O1$ is the circumcenter of triangle $TaTbTc$,
- Se is the External similitude center of circles c and $c1$,
- Si is the Internal similitude center of circles c and $c1$.

SUPPLEMENTARY MATERIAL

The enclosed supplementary material contains theorems related to the topic.

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