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Sums With Square Distances Between a Point and Vertexes

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Abstract. In this paper we study the locus of the points P for which the sum $a^n p_a^2 + b^n p_b^2 + c^n p_c^2 = const$, where p_a, p_b, p_c are distances between the point P and the vertexes of the triangle ABC. We make generalization for the sum $kp_a^2 + lp_b^2 + mp_c^2 = const$.

Keywords. Euclidean geometry, triangle geometry, barycentric coordinates, locus of points, circle.

1. Introduction

In this paper we study the locus of the points P for which the sum

$$\sigma(n) = a^n p_a^2 + b^n p_b^2 + c^n p_c^2 = const,$$

where $p_a = PA, p_b = PB, p_c = PC$ are distances between the point P with homogeneous barycentric coordinates $(u : v : w), u + v + w \neq 0$, and the vertexes of the triangle ABC.

2. Preliminaries

We shall work with homogeneous barycentric coordinates. We consider a nondegenerate triangle ABC as the reference triangle, and set up a coordinate system for points in the plane of the triangle.

$$A = (1 : 0 : 0) , \quad B = (0 : 1 : 0) , \quad C = (0 : 0 : 1)$$

We shall make use of John H. Conway's notations [3, §3.4.1]. Let S denote *twice* the area of triangle ABC. For a real number θ, denote $S_\theta = S \cot \theta$. In particular,

$$S_A = \frac{b^2 + c^2 - a^2}{2} , \quad S_B = \frac{c^2 + a^2 - b^2}{2} , \quad S_C = \frac{a^2 + b^2 - c^2}{2}$$

$$S_B + S_C = a^2 , \quad S_C + S_A = b^2 , \quad S_A + S_B = c^2$$

$$S_{AB} = S_A S_B , \quad S_{BC} = S_B S_C , \quad S_{CA} = S_C S_A$$

$$S^2 = S_{AB} + S_{BC} + S_{CA} = \frac{1}{4}(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)$$

Definition 1. [3, §7.1] **The distance formula in homogeneous barycentric coordinates**

If $P = (x : y : z)$ and $Q = (u : v : w)$, the **square distance** between P and Q is given by:

$$|PQ|^2 = \frac{1}{(u+v+w)^2(x+y+z)^2} \sum_{cyclic} S_A((v+w)x - u(y+z))^2$$

Definition 2. [3, §4.4] **Pedal triangle**

The **pedals** P_a, P_b, P_c of a point $P = (u : v : w), u + v + w \neq 0$ are the intersections of the sidelines with the corresponding perpendiculars through P . The triangle $P_aP_bP_c$ is called the **pedal triangle** of triangle ABC and P .

$$\begin{pmatrix} P_a \\ P_b \\ P_c \end{pmatrix} = \begin{pmatrix} 0 & S_Cu + a^2v & S_Bu + a^2w \\ S_Cv + b^2u & 0 & S_Av + b^2w \\ S_Bw + c^2u & S_Aw + c^2v & 0 \end{pmatrix} \tag{1}$$

Definition 3. [3, §7.2.1] **Circle with radius** ρ

The equation of the circle with center $(u : v : w)$ and radius ρ is:

$$a^2yz + b^2zx + c^2xy - (x+y+z) \sum_{cyclic} \left(\frac{c^2v^2 + 2S_Avw + b^2w^2}{(u+v+w)^2} - \rho^2 \right) x = 0$$

3. Square distances

The square distances between P and the vertexes of triangle ABC are:

$$\begin{aligned} p_a^2 &= \frac{S_Bv^2 + S_Cw^2 + S_A(v+w)^2}{(u+v+w)^2} = \frac{c^2v^2 + b^2w^2 + 2S_Avw}{(u+v+w)^2} \\ p_b^2 &= \frac{S_Cw^2 + S_Au^2 + S_B(w+u)^2}{(u+v+w)^2} = \frac{a^2w^2 + c^2u^2 + 2S_Bwu}{(u+v+w)^2} \\ p_c^2 &= \frac{S_Au^2 + S_Bv^2 + S_C(u+v)^2}{(u+v+w)^2} = \frac{b^2u^2 + a^2v^2 + 2S_Cuv}{(u+v+w)^2} \end{aligned} \tag{2}$$

4. $\sigma(0) = p_a^2 + p_b^2 + p_c^2$

$$\begin{aligned} \sigma(0) &= p_a^2 + p_b^2 + p_c^2 \\ &= \frac{S_B(2v^2 + (w+u)^2) + S_C(2w^2 + (u+v)^2) + S_A(2u^2 + (v+w)^2)}{(u+v+w)^2} \\ &= \frac{(b^2+c^2)u^2 + (c^2+a^2)v^2 + (a^2+b^2)w^2 + 2S_Avw + 2S_Bwu + 2S_Cuv}{(u+v+w)^2} \end{aligned}$$

When $P = G = (1 : 1 : 1)$, the centroid[2] of ABC :

$$\sigma_G(0) = \frac{1}{3}(a^2 + b^2 + c^2)$$

[2]The centroid appears in ETC [2] as the point X_2 .

Theorem 1. *Let $\mathcal{C}_{\mathcal{G}\rho}$ be a circle with center G and radius ρ. For the points of the circle $\mathcal{C}_{\mathcal{G}\rho}$, $\sigma(0) = const$.*

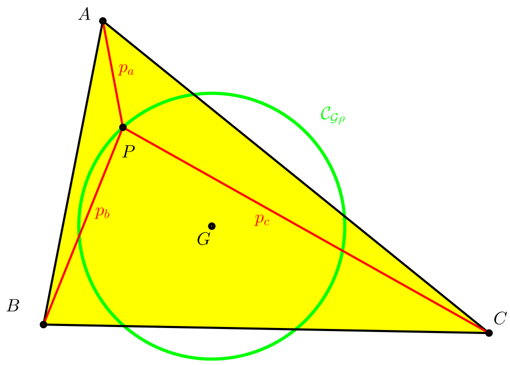

FIGURE 1.

Proof. .
The equation of the circle $\mathcal{C}_{\mathcal{G}\rho}$ is:

$$a^2yz+b^2xz+c^2xy-\frac{1}{9}(x+y+z)((2S_A+b^2+c^2)x+(2S_B+c^2+a^2)y+(2S_C+a^2+b^2)z-9\rho^2(x+y+z)) = 0$$

For point $P = (u : v : w) \in \mathcal{C}_{\mathcal{G}\rho}$, $(\rho = const)$:

$$\frac{1}{9}(a^2-2b^2-2c^2)u^2+\frac{1}{9}(-2a^2+b^2-2c^2)v^2+\frac{1}{9}(-2a^2-2b^2+c^2)w^2+\frac{1}{9}(5a^2-b^2-c^2)vw$$
$$+\frac{1}{9}(-a^2-b^2+5c^2)uv+\frac{1}{9}(-a^2+5b^2-c^2)wu+\rho^2(u+v+w)^2 = 0$$

Or

$$\frac{1}{9}(a^2+b^2+c^2+9\rho^2)(u+v+w)^2$$
$$-\frac{3((b^2+c^2)u^2+(c^2+a^2)v^2+(a^2+b^2)w^2+2S_Avw+2S_Bwu+2S_Cuv)}{9} = 0$$
$$\frac{1}{9}(a^2+b^2+c^2+9\rho^2)(u+v+w)^2-\frac{1}{3}\sigma(0).(u+v+w)^2 = 0$$

$$\sigma(0) = \frac{1}{3}(a^2+b^2+c^2+9\rho^2) = \sigma_G(0)+3\rho^2 = const.$$

(See [1, Theorem 5.17, p. 110]) □

Corollary 1. *For distances $p_a = PA, p_b = PB, p_c = PC$ between point P and the vertexes of the triangle ABC is valid*

$$p_a^2+p_b^2+p_c^2 \geqslant \frac{1}{3}(a^2+b^2+c^2)$$

with equality if and only if $P = G$.

5. $\sigma(1) = ap_a^2+bp_b^2+cp_c^2$

$$\sigma(1) = ap_a^2+bp_b^2+cp_c^2 = \frac{bc(b+c)u^2+ca(c+a)v^2+ab(a+b)w^2+2aS_Avw+2bS_Bwu+2cS_Cuv}{(u+v+w)^2}$$

When $P = I = (a : b : c)$, the incenter[3] of ABC :

$$\sigma_I(1) = abc$$

Theorem 2. *Let $\mathcal{C}_{\mathcal{I}\rho}$ be a circle with center I and radius ρ. For the points of the circle $\mathcal{C}_{\mathcal{I}\rho}$, $\sigma(1) = const.$*

Proof. . The equation of the circle $\mathcal{C}_{\mathcal{I}\rho}$ is:

$$a^2yz+b^2xz+c^2xy-\frac{(x+y+z)}{a+b+c}(bc(-a+b+c)x+ca(a-b+c)y+ab(a+b-c)z-(a+b+c)(x+y+z)\rho^2)=0$$

For point $P = (u : v : w) \in \mathcal{C}_{\mathcal{I}\rho}, (\rho = const)$:

$$\begin{aligned}
&\frac{1}{a+b+c}(bc(a-b-c)u^2+ca(-a+b-c)v^2+ab(-a-b+c)w^2\\
&+a(a^2-(b-c)^2)vw+c(c^2-(a-b)^2)uv+b(b^2-(c-a)^2)wu)\\
&+(a+b+c)(u+v+w)^2\rho^2)=0\\
&\text{Or}\\
&bc(b+c)u^2+ca(c+a)v^2+ab(a+b)w^2+2aS_Avw+2bS_Bwu+2cS_Cuv\\
&-(abc+(a+b+c)\rho^2)(u+v+w)^2=0\\
&\sigma(1)(u+v+w)^2-(abc+(a+b+c)\rho^2)(u+v+w)^2=0
\end{aligned}$$

$$\sigma(1) = abc + (a+b+c)\rho^2 = \sigma_I(1) + (a+b+c)\rho^2 = const. \tag{3}$$

□

Corollary 2. *For distances $p_a = PA, p_b = PB, p_c = PC$ between point P and the vertexes of the triangle ABC is valid*

$$ap_a^2 + bp_b^2 + cp_c^2 \geqslant abc$$

with equality if and only if $P = I$.

6. $\sigma(2) = a^2p_a^2 + b^2p_b^2 + c^2p_c^2$

$$\begin{aligned}
\sigma(2) &= a^2p_a^2 + b^2p_b^2 + c^2p_c^2\\
&= \frac{2(b^2c^2u^2 + a^2c^2v^2 + a^2b^2w^2 + a^2S_Avw + b^2S_Bwu + c^2S_Cuv)}{(u+v+w)^2}
\end{aligned} \tag{4}$$

When $P = K = (a^2 : b^2 : c^2)$, the symmedian center[4] of ABC :

$$\sigma_K(2) = \frac{3a^2b^2c^2}{a^2+b^2+c^2}$$

Theorem 3. *Let $\mathcal{C}_{\mathcal{K}\rho}$ be a circle with center K and radius ρ. For the points of the circle $\mathcal{C}_{\mathcal{K}\rho}$, $\sigma(2) = const.$*

Proof. . The equation of the circle $\mathcal{C}_{\mathcal{K}\rho}$ is:

$$\begin{aligned}
a^2yz+b^2xz+c^2xy-\frac{(x+y+z)}{(a^2+b^2+c^2)^2}&(b^2c^2(-a^2+2b^2+2c^2)x+c^2a^2(2a^2-b^2+2c^2)y\\
&+a^2b^2(2a^2+2b^2-c^2)z-(a^2+b^2+c^2)^2(x+y+z)\rho^2)=0
\end{aligned}$$

[3] The incenter appears in ETC [2] as the point X_1 .

[4] The symmedian center appears in ETC [2] as the point X_6 .

Or

$$b^2c^2(-2a^2-2b^2-2c^2+3a^2)x^2+c^2a^2(-2a^2-2b^2-2c^2+3b^2)y^2+a^2b^2(-2a^2-2b^2-2c^2+3c^2)z^2$$
$$+a^2(a^4-b^4-2b^2c^2-c^4+6b^2c^2)yz+b^2(-a^4+b^4-2a^2c^2-c^4+6a^2c^2)zx$$
$$+c^2(-a^4-2a^2b^2-b^4+c^4+6a^2b^2)xy+\rho^2(a^2+b^2+c^2)^2(x+y+z)^2=0$$

$$2(a^2+b^2+c^2)(b^2c^2x^2+c^2a^2y^2+a^2b^2z^2+a^2S_Ayz+b^2S_Bzx+c^2S_Cxy)$$
$$-3a^2b^2c^2(x+y+z)^2-\rho^2(a^2+b^2+c^2)^2(x+y+z)^2=0$$

For point $P=(u:v:w)\in\mathcal{C}_{\mathcal{K}\rho},(\rho=const)$:

$$2(a^2+b^2+c^2)(b^2c^2u^2+c^2a^2v^2+a^2b^2w^2+a^2S_Avw+b^2S_Bwu+c^2S_Cuv)$$
$$-3a^2b^2c^2(u+v+w)^2-\rho^2(a^2+b^2+c^2)^2(u+v+w)^2=0$$

By (4):

$$(a^2+b^2+c^2)\sigma(2).(u+v+w)^2-3a^2b^2c^2(u+v+w)^2-\rho^2(a^2+b^2+c^2)^2(u+v+w)^2=0$$

$$(u+v+w)^2(\sigma(2).(a^2+b^2+c^2)-3a^2b^2c^2-\rho^2(a^2+b^2+c^2)^2)=0$$

$$\sigma(2)=\frac{3a^2b^2c^2}{a^2+b^2+c^2}+(a^2+b^2+c^2)\rho^2=\sigma_K(2)+(a^2+b^2+c^2)\rho^2=const.$$

□

Corollary 3. *For distances $p_a=PA,p_b=PB,p_c=PC$ between point P and the vertexes of the triangle ABC is valid*

$$a^2p_a^2+b^2p_b^2+c^2p_c^2\geqslant\frac{3a^2b^2c^2}{a^2+b^2+c^2}$$

with equality if and only if $P=K$.

7. GENERALIZATION — $\sigma=kp_a^2+lp_b^2+mp_c^2$

$$\sigma=kp_a^2+lp_b^2+mp_c^2$$
$$=\frac{(b^2m+c^2l)u^2+(c^2k+a^2m)v^2+(a^2l+b^2k)w^2+2kS_Avw+2lS_Bwu+2mS_Cuv}{(u+v+w)^2}$$

7.1. $k+l+m\neq0$**.** When $P=Q$ with homogeneous barycentric coordinates $(k:l:m)$,

$$\sigma_Q=\frac{a^2lm+b^2mk+c^2kl}{k+l+m} \tag{5}$$

Theorem 4. *Let $\mathcal{C}_{\mathcal{Q}\rho}$ be a circle with center $Q=(k:l:m),k+l+m\neq0$ and radius ρ. For the points of the circle $\mathcal{C}_{\mathcal{Q}\rho}$, $\sigma=const$.*

Proof. . The equation of the circle $\mathcal{C}_{\mathcal{Q}\rho}$ is:

(6)

$$a^2yz+b^2xz+c^2xy-\frac{(x+y+z)}{(k+l+m)^2}((b^2m^2+c^2l^2+2lmS_A)x+(c^2k^2+a^2m^2+2kmS_B)y$$
$$+(a^2l^2+b^2k^2+2klS_C)z-(k+l+m)^2(x+y+z)\rho^2)=0$$

In other words,

$$(-c^2l^2 + a^2lm - b^2lm - c^2lm - b^2m^2)x^2 + (-c^2k^2 - a^2km + b^2km - c^2km - a^2m^2)y^2$$
$$+ (-b^2k^2 - a^2kl - b^2kl + c^2kl - a^2l^2)z^2$$
$$+ (a^2k^2 - b^2k^2 - c^2k^2 + a^2kl - b^2kl + c^2kl + a^2km + b^2km - c^2km + 2a^2lm)yz$$
$$+ (-a^2kl + b^2kl + c^2kl - a^2l^2 + b^2l^2 - c^2l^2 + 2b^2km + a^2lm + b^2lm - c^2lm)zx$$
$$+ (2c^2kl - a^2km + b^2km + c^2km + a^2lm - b^2lm + c^2lm - a^2m^2 - b^2m^2 + c^2m^2)xy$$
$$+ (k + l + m)^2(x + y + z)^2\rho^2 = 0$$

For point $P = (u : v : w) \in \mathcal{C}_{\mathcal{Q}\rho}, (\rho = const)$:

$$((a^2lm + b^2mk + c^2kl) - ((b^2m + c^2l)(k + l + m)))u^2$$
$$+ ((a^2lm + b^2mk + c^2kl) - ((c^2k + a^2m)(k + l + m)))v^2$$
$$+ ((a^2lm + b^2mk + c^2kl) - ((a^2l + b^2k)(k + l + m)))w^2$$
$$+ 2((a^2lm + b^2mk + c^2kl) - S_Ak(k + l + m))vw$$
$$+ 2((a^2lm + b^2mk + c^2kl) - S_Bl(k + l + m))wu$$
$$+ 2((a^2lm + b^2mk + c^2kl) - S_Cm(k + l + m))uv$$
$$+ (k + l + m)^2(u + v + w)^2\rho^2 = 0$$

$$(a^2lm + b^2mk + c^2kl)(u + v + w)^2$$
$$- ((b^2m + c^2l)u^2 + (c^2k + a^2m)v^2 + (a^2l + b^2k)w^2 + 2kS_Avw + 2lS_Bwu + 2mS_Cuv).(k + l + m)$$
$$+ (k + l + m)^2(u + v + w)^2\rho^2) = 0$$

$$(a^2lm + b^2mk + c^2kl)(u + v + w)^2 - \sigma.(k + l + m)(u + v + w)^2 + (k + l + m)^2(u + v + w)^2\rho^2) = 0$$

$$\sigma = \frac{a^2lm + b^2mk + c^2kl}{k + l + m} + (k + l + m)\rho^2 = \sigma_Q + (k + l + m)\rho^2 = const. \quad (7)$$

□

7.2. $k + l + m = 0$. .

I. $k = l = m = 0$

$\sigma = kp_a^2 + lp_b^2 + mp_c^2 = 0$ for all points P.

II. $k = 0, l = -m \neq 0$

The infinite point of the sideline BC is $(0 : -1 : 1) = (0 : -m : m) = (k : l : m)$, see [3, §4.2.1].

Let D be a point on the sideline BC, and $d = BD$ is signed length of the segment BD.

$$\sigma_D = -mBD^2 + mCD^2 = m(-d^2 + (c - d)^2) = mc(c - 2d)$$

Let $\mathcal{L}$ be perpendicular from D to BC and P lies on the line $\mathcal{L}$.

$$\sigma = -mBP^2 + mCP^2 = -m(BD^2 + DP^2) + m(CD^2 + DP^2) = \sigma_D = const$$

for all points of the line $\mathcal{L}, (d = const)$.

III. $k \neq 0, l \neq 0, m \neq 0, k + l + m = 0$

Lemma 1. *Let l be line through $G = (1 : 1 : 1)$ with infinite point $(k : l : m), k+l+m = 0$. Construct the perpendicular feet A_1, B_1, C_1 of A, B, C on the line l. Prove*
a) $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = 0$;
b) $k\overrightarrow{AA_1} + l\overrightarrow{BB_1} + m\overrightarrow{CC_1} = 0$.

Proof. The point Q with homogeneous barycentric coordinates $(k : l : m), k+l+m = 0$ is infinite point. Parallel lines have the same infinite point. The line l through $G = (1 : 1 : 1)$ with infinite point Q has equation (see [3, §4.2.2]) :

$$\begin{vmatrix} k & l & m \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = 0$$

$$\text{Or} \qquad l : \qquad (l-m)x + (m-k)y + (k-l)z = 0$$

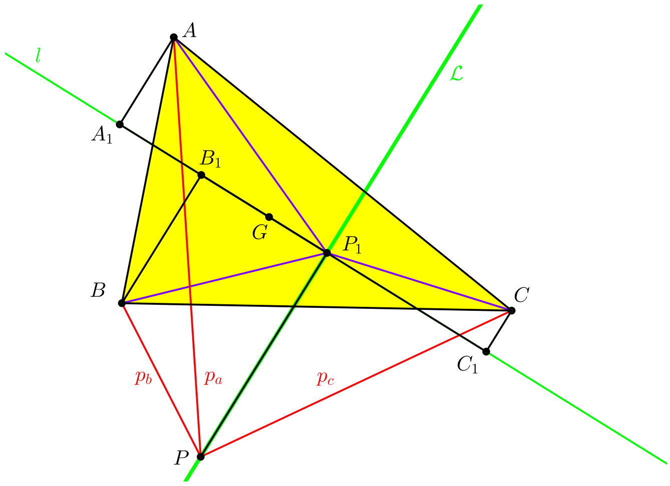

FIGURE 2.

The lines perpendicular to l have infinite point (see [3, §4.5]) :

$$Q' = (S_B l - S_C m : S_C m - S_A k : S_A k - S_B l)$$

The line $\mathcal{L}_A$ through A perpendicular to l has equation

$$\begin{vmatrix} S_B l - S_C m & S_C m - S_A k & S_A k - S_B l \\ 1 & 0 & 0 \\ x & y & z \end{vmatrix} = 0$$

$$\text{Or} \qquad \mathcal{L}_A : \qquad (S_A k - S_B l)y - (S_C m - S_A k)z = 0$$

Let $A_1 = l \cap \mathcal{L}_A$. In absolute barycentric coordinates is

$$A_1 = \frac{2k^2 S_A + l^2 S_B + m^2 S_C - c^2 kl - b^2 km}{\delta} A + \frac{(l-m)(kS_A - mS_C)}{\delta} B + \frac{(l-m)(lS_B - kS_A)}{\delta} C$$

$$\delta = 2k^2 S_A + 2l^2 S_B + 2m^2 S_C - c^2 kl - b^2 km - a^2 lm$$

$$A_1 - A = \left(\frac{(l-m)(mS_C - lS_B)}{\delta} : \frac{(l-m)(kS_A - mS_C)}{\delta} : \frac{(l-m)(lS_B - kS_A)}{\delta} \right)$$

By analogy:

$$B_1 - B = \left(\frac{(m-k)(mS_C - lS_B)}{\delta} : \frac{(m-k)(kS_A - mS_C)}{\delta} : \frac{(m-k)(lS_B - kS_A)}{\delta}\right)$$

$$C_1 - C = \left(\frac{(k-l)(mS_C - lS_B)}{\delta} : \frac{(k-l)(kS_A - mS_C)}{\delta} : \frac{(k-l)(lS_B - kS_A)}{\delta}\right)$$

Hence

$$\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = (A_1 - A) + (B_1 - B) + (C_1 - C) = 0$$

$$k\overrightarrow{AA_1} + l\overrightarrow{BB_1} + m\overrightarrow{CC_1} = k(A_1 - A) + l(B_1 - B) + m(C_1 - C) = 0$$

□

Remark. The condition from Lemma 1 a) is independent from k, l, m and it is true for all lines through G. See [1, Theorem 5.19, p. 112]

Theorem 5. *Let l be line through $G = (1 : 1 : 1)$ with infinite point $(k : l : m), k + l + m = 0$. Let $\mathcal{L}$ is a line perpendicular to l. For all points P of the line $\mathcal{L}$, $\sigma = kp_a^2 + lp_b^2 + mp_c^2 = const.$*

Proof. Let the point P lies on the line $\mathcal{L}$ and $P_1 = l \cap \mathcal{L}$. See Figure 2. Let $\sigma_1 = kP_1A^2 + lP_1B^2 + mP_1C^2$.

$$\begin{aligned}\sigma_1 &= kP_1A^2 + lP_1B^2 + mP_1C^2 \\ &= k(P_1A_1^2 + AA_1^2) + l(P_1B_1^2 + BB_1^2) + m(P_1C_1^2 + CC_1^2)\end{aligned}$$

$$\begin{aligned}\sigma &= kPA^2 + lPB^2 + mPC^2 \\ &= k(P_1A_1^2 + (\overrightarrow{AA_1} + \overrightarrow{P_1P})^2) + l(P_1B_1^2 + (\overrightarrow{BB_1} + \overrightarrow{P_1P})^2) + m(P_1C_1^2 + (\overrightarrow{CC_1} + \overrightarrow{P_1P})^2) \\ &= \sigma_1 + (k + l + m)\overrightarrow{P_1P}^2 + 2\overrightarrow{P_1P}(k\overrightarrow{AA_1} + l\overrightarrow{BB_1} + m\overrightarrow{CC_1})\end{aligned}$$

By Lemma 1: $k\overrightarrow{AA_1} + l\overrightarrow{BB_1} + m\overrightarrow{CC_1} = 0$. Hence

$$\sigma = \sigma_1 = const.$$

□

8. $\sigma(n) = a^n p_a^2 + b^n p_b^2 + c^n p_c^2$

$$\begin{aligned}\sigma(n) &= a^n p_a^2 + b^n p_b^2 + c^n p_c^2 \\ &= \frac{(b^2c^n + c^2b^n)u^2 + (c^2a^n + a^2c^n)v^2 + (a^2b^n + b^2a^n)w^2 + 2a^nS_Avw + 2b^nS_Bwu + 2c^nS_Cuv}{(u+v+w)^2}\end{aligned}$$

When P has homogeneous barycentric coordinates $(a^n : b^n : c^n)$, see (5),

$$\sigma_Q(n) = \frac{a^2b^nc^n + b^2c^na^n + c^2a^nb^n}{a^n + b^n + c^n}$$

Theorem 6. *Let $\mathcal{C}_{Q\rho}$ be a circle with center $Q = (a^n : b^n : c^n)$ and radius ρ. For the points of the circle $\mathcal{C}_{Q\rho}$, $\sigma(n) = const.$*

Proof. . The equation of the circle $\mathcal{C}_{Q\rho}$ is , see (6):

$$\begin{aligned}a^2yz + b^2xz + c^2xy - \frac{(x+y+z)}{(a^n+b^n+c^n)^2}&((b^2c^{2n} + c^2b^{2n} + 2b^nc^nS_A)x + (c^2a^{2n} + a^2c^{2n} + 2a^nc^nS_B)y \\ &+ (a^2b^{2n} + b^2a^{2n} + 2a^nb^nS_C)z - (a^n + b^n + c^n)^2(x+y+z)\rho^2) = 0\end{aligned}$$

For point $P = (u : v : w) \in \mathcal{C}_{Q\rho}, (\rho = const)$, see (7) :

$$\sigma(n) = \frac{a^2b^nc^n + b^2c^na^n + c^2a^nb^n}{a^n + b^n + c^n} + (a^n+b^n+c^n)\rho^2 = \sigma_Q(n) + (a^n+b^n+c^n)\rho^2 = const.$$

□

Corollary 4. *For distances $p_a = PA, p_b = PB, p_c = PC$ between point P and the vertexes of the triangle ABC is valid*

$$a^np_a^2 + b^np_b^2 + c^np_c^2 \geqslant \frac{a^2b^nc^n + b^2c^na^n + c^2a^nb^n}{a^n + b^n + c^n}$$

with equality if and only if P has homogeneous barycentric coordinates $(a^n : b^n : c^n)$.

9. PEDAL TRIANGLE

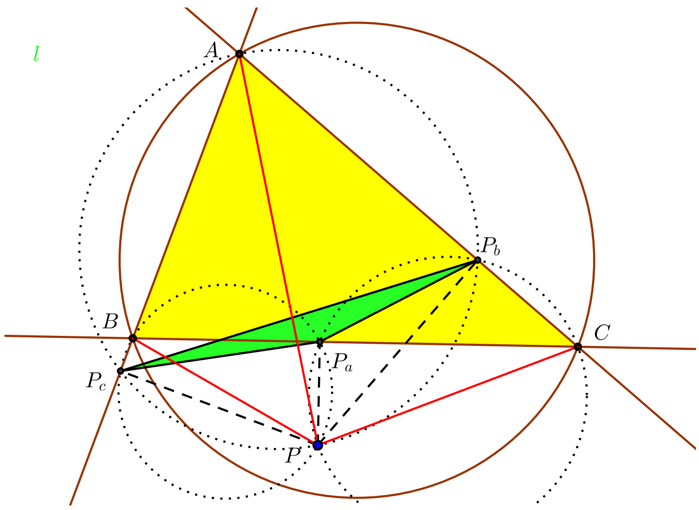

FIGURE 3.

Theorem 7. *Let $k + l + m \neq 0$. Let $\mathcal{C}_{Q\rho}$ be a circle with center $Q = (k : l : m)$ and radius ρ. For the points P of the circle $\mathcal{C}_{Q\rho}$,*

$$\frac{k}{a^2}{P_bP_c}^2 + \frac{l}{b^2}{P_cP_a}^2 + \frac{m}{c^2}{P_aP_b}^2 = const.$$

Proof. The quadrilaterals $PP_bAP_c, PP_cBP_a, PP_aCP_b$ are inscribed quadrilaterals with diameters of the circles PA, PB, PC respectively, Figure 3. By the law of sines applying to triangles $AP_bP_c, BP_cP_a, CP_aP_b$, we have

$$P_bP_c = PAsinA\ , \quad P_cP_a = PBsinB\ , \quad P_aP_b = PCsinC$$

Applying the law of sines to triangle ABC, we have

$$sinA = \frac{a}{2R}\ , \quad sinB = \frac{b}{2R}\ , \quad sinC = \frac{c}{2R}$$

It follows that

$$P_bP_c = \frac{aPA}{2R}\ , \quad P_cP_a = \frac{bPB}{2R}\ , \quad P_aP_b = \frac{cPC}{2R}$$

$$\frac{k}{a^2}{P_bP_c}^2 + \frac{l}{b^2}{P_cP_a}^2 + \frac{m}{c^2}{P_aP_b}^2 = \frac{kPA^2 + lPB^2 + mPC^2}{4R^2}$$

$$= \frac{a^2lm + b^2mk + c^2kl}{4R^2(k+l+m)} + \frac{k+l+m}{4R^2}\rho^2 = const\ . \quad \text{(see Theorem 4)}$$

□

Theorem 8. *Let $k+l+m=0$. Let l be line through $G=(1:1:1)$ with infinite point $(k:l:m)$. Let $\mathcal{L}$ is a line perpendicular to l. For all points P of the line $\mathcal{L}$*

$$\frac{k}{a^2}{P_bP_c}^2+\frac{l}{b^2}{P_cP_a}^2+\frac{m}{c^2}{P_aP_b}^2=const.$$

Proof. By Theorem 5 follows

$$\begin{aligned}&\frac{k}{a^2}{P_bP_c}^2+\frac{l}{b^2}{P_cP_a}^2+\frac{m}{c^2}{P_aP_b}^2\\ &=\frac{kPA^2+lPB^2+mPC^2}{4R^2}=\frac{\sigma}{4R^2}=const.\end{aligned}$$

□

10. Examples

10.1. **Incircle.**

Example 1. *Let P be a point on incircle of triangle ABC with S – twice the area of triangle. Then*

$$a.PA^2+b.PB^2+c.PC^2=abc+\frac{S^2}{a+b+c}$$

Proof. It is known that the inradius $r=\frac{S}{(a+b+c)}$. From (3)

$$\sigma(1)=abc+(a+b+c)r^2=abc+\frac{S^2}{a+b+c}$$

□

10.2. **Circumcircle.**

Example 2. *Let P be a point on the circumcircle of triangle ABC. Then sum $a^2S_Ap_a^2+b^2S_Bp_b^2+c^2S_Cp_c^2=a^2b^2c^2$.*

Proof. It is known that the circumcenter[5] has homogeneous barycentric coordinates $O=(a^2S_A:b^2S_B:c^2S_C)$ (See [3, §3.1.1]) and circumradius $R=\frac{abc}{2S}$. From (7):

$$\begin{aligned}&a^2S_Ap_a^2+b^2S_Bp_b^2+c^2S_Cp_c^2\\ &=\frac{a^2b^2c^2(S_BS_C+S_CS_A+S_AS_B)}{a^2S_A+b^2S_B+c^2S_C}+(a^2S_A+b^2S_B+c^2S_C)R^2\\ &=\frac{a^2b^2c^2S^2}{2S^2}+2S^2\frac{(abc)^2}{4S^2}\\ &=a^2b^2c^2\end{aligned}$$

□

[5]The circumcenter appears in ETC [2] as the point X_3 .

10.3. **Euler line.**

Example 3. *Let P be a point on the Euler line of triangle ABC. The sum* $\sigma = (b^2-c^2)p_a^2 + (c^2-a^2)p_b^2 + (a^2-b^2)p_c^2 = 0$.

Proof. The line containing O, G, H is called the Euler line of triangle ABC .
The equation of the Euler line $\mathcal{L}_{\mathcal{E}}$, as the line joining the centroid $G = (1:1:1)$ to the orthocenter $H = (S_{BC} : S_{CA} : S_{AB})$ is (see [3, §4.1.2])

$$(S_{AB}-S_{CA})x + (S_{BC}-S_{AB})y + (S_{CA}-S_{BC})z = 0$$

The infinite point of the Euler line is the point $X_{30} = (3S_{BC}-S^2 : 3S_{CA}-S^2 : 3S_{AB}-S^2)$. The infinite point of the line l perpendicular to Euler line is $(S_C-S_B : S_A-S_C : S_B-S_C) = (b^2-c^2 : c^2-a^2 : a^2-b^2)$.
For point O, $\sigma = (b^2-c^2)R^2 + (c^2-a^2)R^2 + (a^2-b^2)R^2 = 0$.
For point G,

$$\begin{aligned}\sigma &= (b^2-c^2)\left(\frac{2}{3}m_a\right)^2 + (c^2-a^2)\left(\frac{2}{3}m_b\right)^2 + (a^2-b^2)\left(\frac{2}{3}m_c\right)^2\\ &= \frac{1}{9}((b^2-c^2)(2b^2+2c^2-a^2) + (c^2-a^2)(2c^2+2a^2-b^2)\\ &+ (a^2-b^2)(2a^2+2b^2-c^2)) = 0\end{aligned}$$

According Theorem 5 the sum $\sigma = (b^2-c^2)p_a^2 + (c^2-a^2)p_b^2 + (a^2-b^2)p_c^2 = const = 0$ for all points P from the Euler line $\mathcal{L}_{\mathcal{E}}$. □

10.4. **Same inequalities.** See corollary 4

n	Center Q	$\sigma(n) = a^n p_a^2 + b^n p_b^2 + c^n p_c^2$
4	X_{32}	$\sigma(4) \geqslant \frac{a^2b^2c^2(a^2b^2+b^2c^2+c^2a^2)}{a^4+b^4+c^4}$
3	X_{31}	$\sigma(3) \geqslant \frac{a^2b^2c^2(ab+bc+ca)}{a^3+b^3+c^3}$
2	$K = X_6$	$\sigma(2) \geqslant \frac{3a^2b^2c^2}{a^2+b^2+c^2}$
$\frac{3}{2}$	X_{365}	$\sigma(\frac{3}{2}) \geqslant \frac{a^{3/2}b^{3/2}c^{3/2}(\sqrt{a}+\sqrt{b}+\sqrt{c})}{a^{3/2}+b^{3/2}+c^{3/2}}$
1	$I = X_1$	$\sigma(1) \geqslant abc$
$\frac{1}{2}$	X_{366}	$\sigma(\frac{1}{2}) \geqslant \frac{\sqrt{abc}(a^{3/2}+b^{3/2}+c^{3/2})}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$
0	$G = X_2$	$\sigma(0) \geqslant \frac{1}{3}(a^2+b^2+c^2)$
-1	X_{75}	$\sigma(-1) \geqslant \frac{a^3+b^3+c^3}{ab+bc+ca}$
-2	X_{76}	$\sigma(-2) \geqslant \frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2}$

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