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On Some Properties Of Neuberg Cubic

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Abstract. By using the computer program “Mathematica”, we give theorems about Neuberg Cubic.

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Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The Neuberg cubic K001 is introduced in Neuberg's paper "Mémoire sur le tétraèdre" in Mémoires de l'Académie de Belgique, pp.1–70, 1884. The characterization given is related with properties of so-called "quadrangles involutifs":

$$M \in K001 \Leftrightarrow \begin{vmatrix} 1 & BC^2 + AM^2 & BC^2 \times AM^2 \\ 1 & CA^2 + BM^2 & CA^2 \times BM^2 \\ 1 & AB^2 + CM^2 & AB^2 \times CM^2 \end{vmatrix} = 0$$

It passes through Kimberling centers X_i for $i = 1$ (incenter I), 3 (circumcenter O), 4 (orthocenter H), 13 (first Fermat point), 14 (second Fermat point), 15 (first isodynamic point), 16 (second isodynamic point), 30 (Euler infinity point), 74, in addition to 399 (Parry reflection point), 484 (first Evans perspector), $616, 617, 1138, 1157, 1263, 1276$ (second Evans perspector), 1277 (third Evans perspector), $2118, 2132$ and 2133. It also passes through excenters J_A, J_B, and J_C of the reference triangle and the circular points at infinity.

K001 is sometimes called 21-point cubic or 37-point cubic in older literature. K001 is the isogonal pK with pivot X(30) = infinite point of the Euler line : it is the locus of point P such that the line PP^* is parallel to the Euler line (P^* isogonal conjugate of P). Hence it is a member of the Euler pencil [1].

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In this paper, we shall focus on some properties of Neuberg cubic related to perspective triangles, sub-cubics and circles.

2. Preliminaries

Definition 2.1. *The Steiner circles of a triangle are the circles passing through a vertex and the reflections of the other two vertices on the centroid G. The Steiner circles of a triangle concur at a point called the Steiner point of the triangle [6]. It belongs to the circumcircle. It is the point X_{99} in ETC. It has barycentric coordinates $(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2})$ [2].*

Definition 2.2. *Let L, M and N be lines through A, B, C respectively, parallel to the Euler line. Let $L^{'}$ be the reflection of L in sideline BC, let $M^{'}$ be the reflection of M in sideline CA, and let $N^{'}$ be the reflection of N in sideline AB. The lines $L^{'}$, $M^{'}$, and $N^{'}$ then concur in a point known as the Parry reflection point, which is Kimberling center X_{399} [4]. It has first barycentric coordinates $(a^2(a^8 - 4a^6(b^2+c^2) + a^4(6b^4 + b^2c^2 + 6c^4) - a^2(4b^6 - b^4c^2 - b^2c^4 + 4c^6) + (b^2 - c^2)^2(b^4 + 4b^2c^2 + c^4)) : ... : ...)$ [2].*

Definition 2.3. *Let three similar isosceles triangles $\triangle A^{'}BC$, $\triangle AB^{'}C$, and $\triangle ABC^{'}$ be constructed on the sides of a triangle $\triangle ABC$. Then $\triangle ABC$ and $\triangle A^{'}B^{'}C^{'}$ are perspective triangles, and the envelope of their perspectrix as the vertex angle of the erected triangles is varied is a parabola known as the Kiepert parabola [3]. X_{110} of $\triangle ABC$ is focus of Kiepert Parabola. X_{110} has barycentric coordinates $(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2})$ [2].*

Definition 2.4. *The Kiepert hyperbola is a hyperbola and triangle conic that is related to the solution of Lemoine's problem and its generalization to isosceles triangles constructed on the sides of a given triangle [5]. Center of Kiepert Hyperbola is the point X_{115} in ETC. X_{115} has barycentric coordinates $((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$ [2].*

Definition 2.5. *The Jerabek hyperbola is a circumconic that is the isogonal conjugate of the Euler line [7]. Center of Jerabek Hyperbola is the point X_{125} in ETC. X_{125} has barycentric coordinates $((a^2-b^2-c^2)(b^2-c^2)^2 : -(a^2-c^2)^2(a^2-b^2+c^2) : -(a^2-b^2)^2(a^2+b^2-c^2))$ [2].*

Definition 2.6. *The Napoleon-Feuerbach cubic is the isogonal pK with pivot X_5 = nine-point center [1].*

It passes through Kimberling centers X_i for $i = 1$ (incenter I), 3 (circumcenter O), 4 (orthocenter H), 5 (nine-point center N), 17 (first Napoleon point), 18 (second Napoleon point), 54 (Kosnita point), $61, 62, 195, 627, 628, 2120, 2121$, as well as the excenters J_A, J_B, and J_C [1].

Definition 2.7. *An involution, or an involutory function, is a function f that is its own inverse, $f(f(x)) = x$ for all x in the domain of f. Equivalently, applying f twice produces the original value.*

3. Theorems

Theorem 3.1. *Let P be a point on Neuberg Cubic of ABC. Neuberg cubics of ABC, PBC, PCA and PAB concur at a point Q (Figure 1).*

Some pair of points (P, Q) listed in ETC. (X_1, X_{3065}),(X_3, X_{74}),(X_4, X_{1263}), (X_{13}, X_{14}),(X_{14}, X_{13}),(X_{15}, X_{8546}),(X_{399}, X_{8487}),(X_{484}, X_{7329}),(X_{617}, X_{8492}).

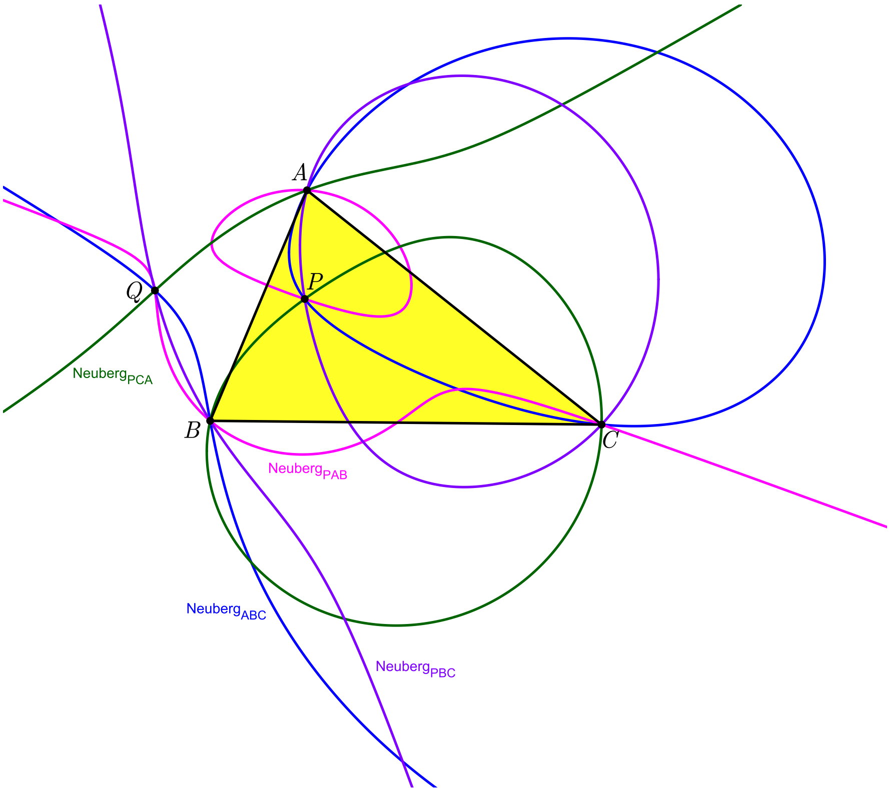

Figure 1. Neuberg Cubics

Theorem 3.2. *Let P be a point on Neuberg Cubic of ABC. Neuberg cubics of ABC, PBC, PCA and PAB concur at a point Q. Lines PQ pass through a fixed point. This is the point X_{399}, Parry-Reflection point of ABC.*

Let's call the transformation $\mathcal{T}$, which maps the point P to point Q i.e. $\mathcal{T}(\mathcal{P}) = Q$.

Theorem 3.3. *Transformation $\mathcal{T}(\mathcal{P})$ is involution. $\mathcal{T}(\mathcal{T}(\mathcal{P}))$=$P$.*

We can get the point Q by perspective triangles.

Theorem 3.4. *Let P be a point on Neuberg Cubic of ABC. D be X_{399} of PBC. Define E, F cyclically. ABC and DEF are perspective and the perspector is the point Q (Figure 2).*

Theorem 3.5. *Let P be a point on Neuberg Cubic of ABC. Napoleon-Feuerbach cubics of ABC, PBC, PCA and PAB concur at a point Q (Figure 3).*

Theorem 3.6. *Let P be a point on Neuberg Cubic of ABC. Napoleon-Feuerbach cubics of ABC, PBC, PCA and PAB concur at a point Q. Lines PQ pass through a fixed point and this point is the X_3, circumcenter of ABC.*

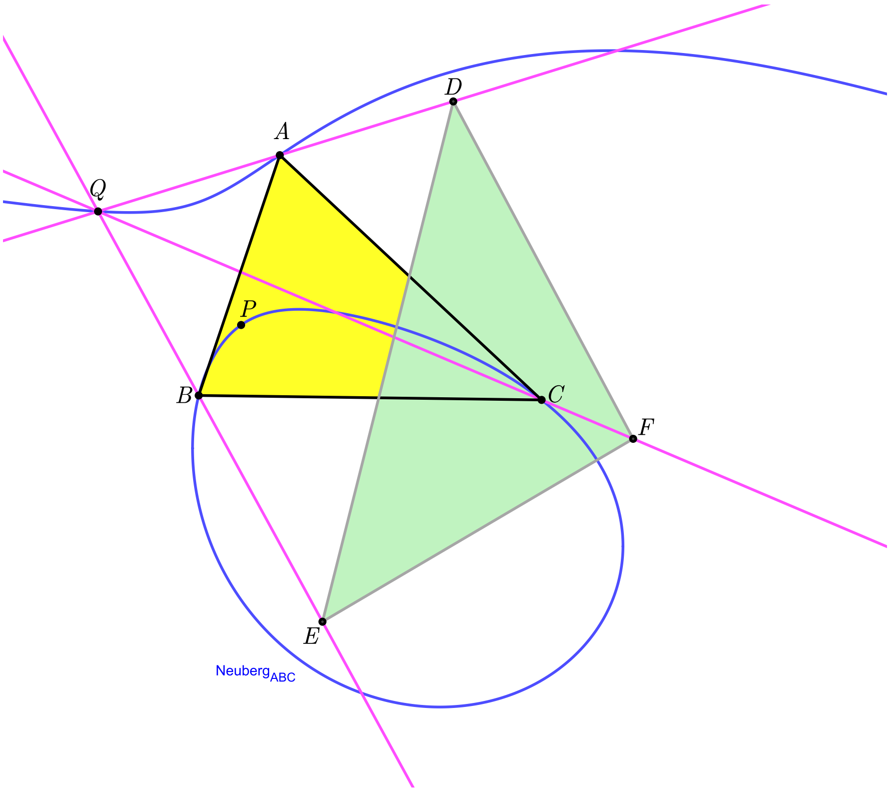

FIGURE 2.

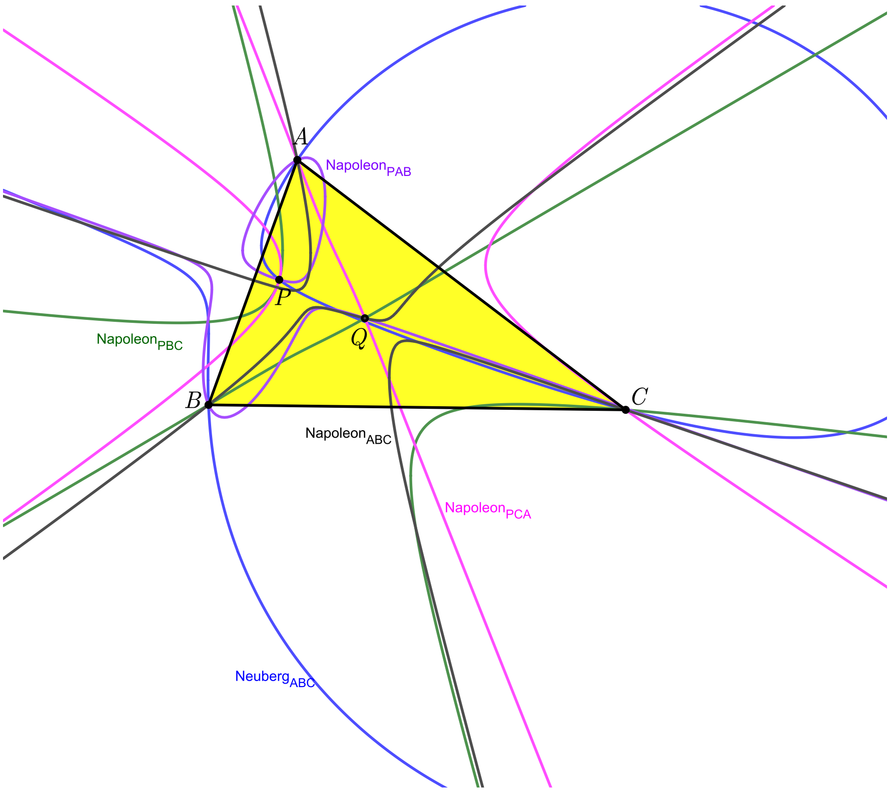

FIGURE 3.

Theorem 3.7. *Let P and P^* be two isogonal points on Neuberg Cubic of ABC. Let Napoleon-Feuerbach cubics of ABC, PBC, PCA and PAB concur at a point*

Q and Napoleon-Feuerbach cubics of ABC, P^*BC, P^*CA and P^*AB concur at a point Q^*. P^* and Q^* are also isogonal conjugate points lying on Napoleon Feuerbach cubic of ABC.

In other words this transformation (Transforming P to Q) preserves the isogonality. Some examples to $(P, P^*) \to (Q, Q^*) : (X_3, X_4) \to (X_{54}, X_5)$, $(X_{13}, X_{15}) \to (X_{17}, X_{61})$, $(X_{14}, X_{16}) \to (X_{18}, X_{62})$,$(X_{399}, X_{1138}) \to (X_{3470}, X_{3471})$.

Theorem 3.8. *Let P be a point on Neuberg Cubic of ABC. S_a ve X_{110} of PBC. Define S_b, S_c clically. Circles (AS_bS_c),(BS_cS_a), (CS_aS_b) pass through a fixed point. It is the point X_{110} of ABC. If O_a, O_b, O_c are centers of these circles respectively, ABC and $O_aO_bO_c$ are perspective at a point X (Figure 4).*

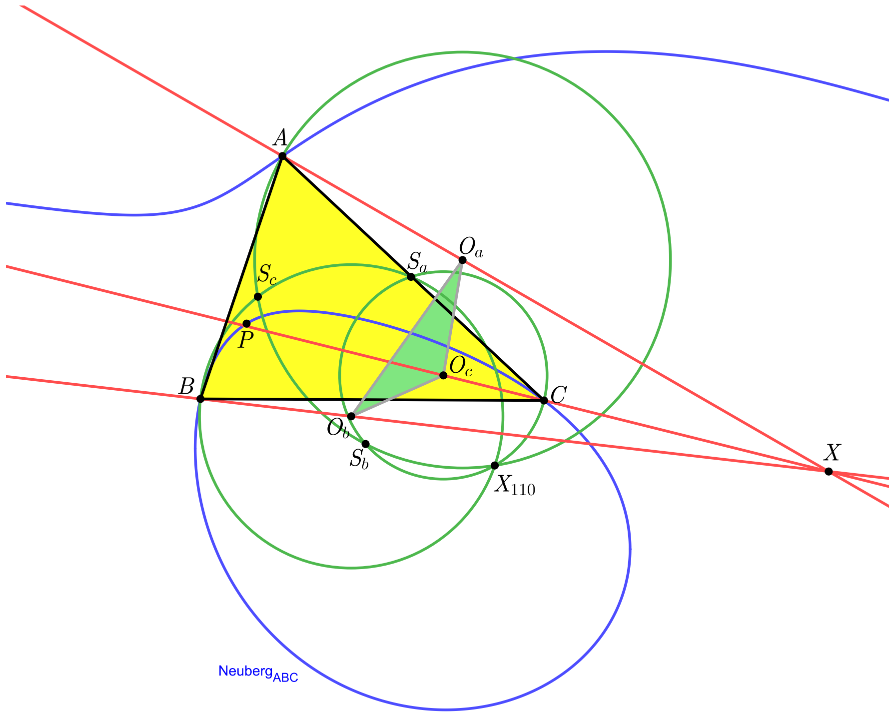

Figure 4.

Some pairs (P, X) where P is a point on Neuberg cubic of ABC and X perspector of ABC and $O_aO_bO_c$: (X_1, X_{36}), (X_{13}, X_{16}), (X_{14}, X_{15}), (X_{15}, X_{6104}), (X_{16}, X_{6105}), (X_{74}, X_3), (X_{399}, X_{14354}). For $P = X_{484}$, X has first $6 - 9 - 13$ search number; $[-8.7122545985646 : ... : ...]$.

Theorem 3.9. *Let P be a point. O_a, circumcenter of PBC. Define O_b, O_c, cyclically. S_a, $X(110)$ of AO_bO_c. Define S_b, S_c cyclically.*

- *S_a, S_b, S_c, and $X(110)$ of ABC lie on same circle if P lies on Neuberg Cubic. Moreover these circles are tangent to circumcircle of ABC at $X(110)$ of ABC.*
- *If $P = X(399)$-Parry reflection point, S_a, S_b, S_c and $X(110)$ of ABC lie on same line. Its the line $X(110)X(351)$ tangent to circumcircle of ABC at $X(110)$. Trilinear pole of this line is $X(249)$ (Figure 5).*

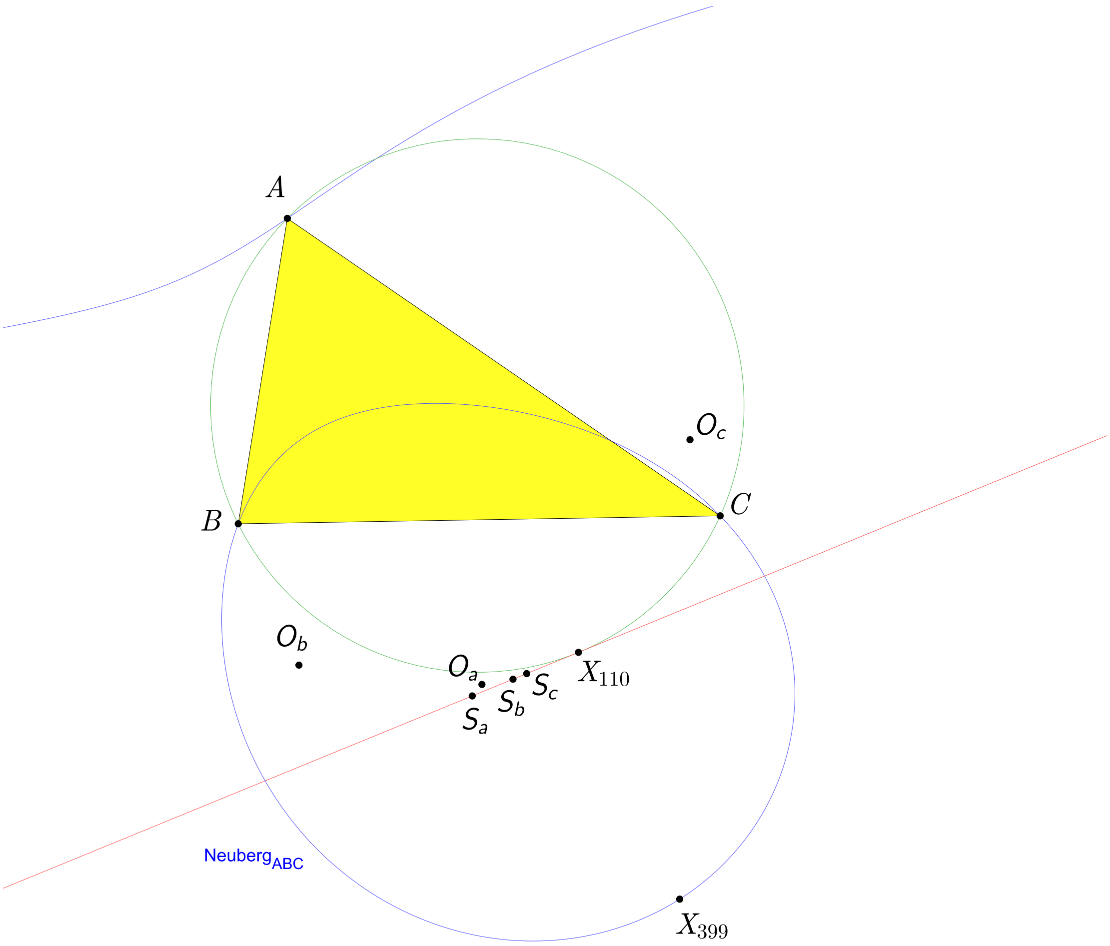

Figure 5.

Theorem 3.10. *Let P be a point on Neuberg Cubic of ABC. $D = X_{99}$-Steiner point of PBC. Define E, F cyclically. P, D, E, F lie on same circle (Figure 6).*

Theorem 3.11. *Let P be a point on Neuberg Cubic of ABC. $D = X_{110}$ of PBC. Define E, F cyclically. P, D, E, F lie on same circle.*

Theorem 3.12. *Let P be a point on Neuberg Cubic of ABC. $D = X_{115}$ of PBC. Define E, F cyclically. X_{115} of ABC, D, E, F lie on same circle.*

Theorem 3.13. *Let P be a point on Neuberg Cubic of ABC. $D = X_{125}$ of PBC. Define E, F cyclically. X_{125} of ABC, D, E, F lie on same circle.*

Theorem 3.14. *Let P be a point on Neuberg Cubic of ABC. DEF be circumcevian triangle of P. $S_a = X_{99}$-Steiner of DBC. Define S_b, S_c cyclically. ABC and DEF are perspective at a point X (Figure 7).*

Some pairs (P, X): (X_1, X_1), (X_3, X_{8884}), (X_4, X_{5562}), (X_{13}, X_{37848}), (X_{14}, X_{37850}), (X_{15}, X_{11581}), (X_{16}, X_{11582}), (X_{74}, X_{16163}). For $P = X_{399}$, X has first $6-9-13$ search number; $[3.7629112261135 : ... : ...]$.

Theorem 3.15. *Let P be a point on Neuberg Cubic of ABC. DEF be circumcevian triangle of P. $S_a = X_{110}$ of DBC. Define S_b, S_c cyclically. ABC and DEF are perspective at a point X.*

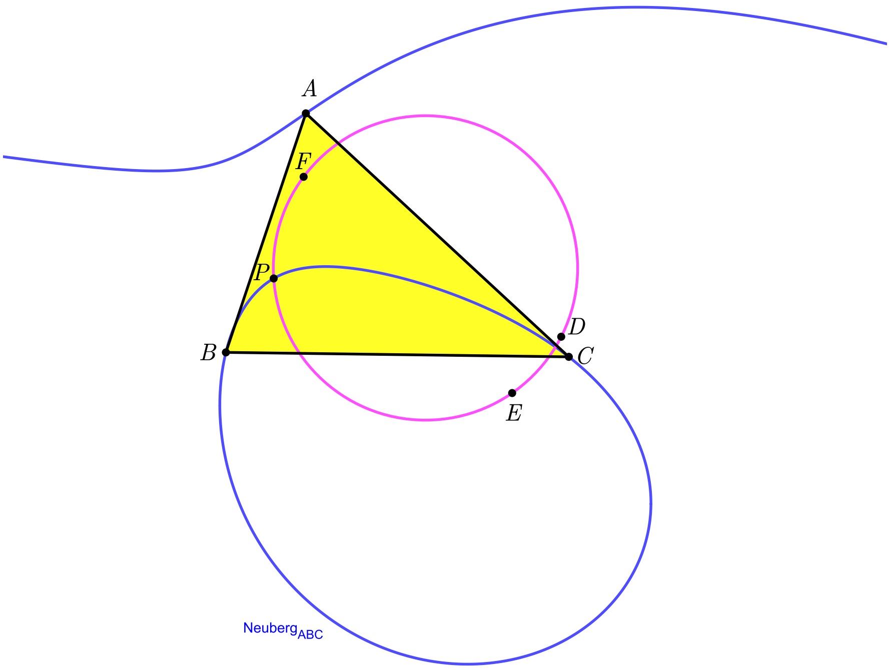

FIGURE 6.

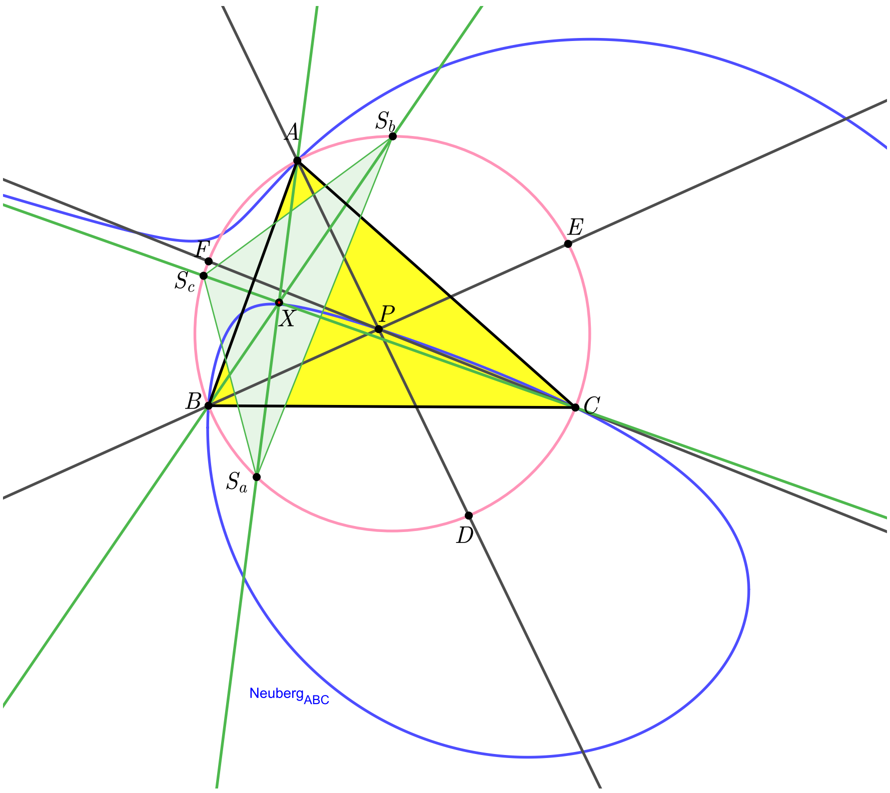

FIGURE 7.

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