

Arrangement of Central Points on the Faces of a Tetrahedron

STANLEY RABINOWITZ

545 Elm St Unit 1, Milford, New Hampshire 03055, USA

e-mail: stan.rabinowitz@comcast.net

web: <http://www.StanleyRabinowitz.com/>

Abstract. We systematically investigate properties of various triangle centers (such as orthocenter or incenter) located on the four faces of a tetrahedron. For each of six types of tetrahedra, we examine over 100 centers located on the four faces of the tetrahedron. Using a computer, we determine when any of 16 conditions occur (such as the four centers being coplanar). A typical result is: The lines from each vertex of a circumscribable tetrahedron to the Gergonne points of the opposite face are concurrent.

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1. INTRODUCTION

Over the centuries, many notable points have been found that are associated with an arbitrary triangle. Familiar examples include: the centroid, the circumcenter, the incenter, and the orthocenter. Of particular interest are those points that Clark Kimberling classifies as “triangle centers”. He notes over 100 such points in his seminal paper [10].

Given an arbitrary tetrahedron and a choice of triangle center (for example, the circumcenter), we may locate this triangle center in each face of the tetrahedron. We wind up with four points, one on each face. What can be said about these points? For example, do the 4 points form a tetrahedron similar to the original one? Could these 4 points ever lie in a plane? Might they form a regular tetrahedron? Consider the 4 lines from the vertices of the tetrahedron to the centers in the opposite faces. Do these 4 lines concur? Might they have the same length?

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In this paper, we investigate such questions for a large collection of triangle centers and for various types of tetrahedra.

A typical result is: The lines from each vertex of a circumscribable tetrahedron to the Gergonne points of the opposite face are concurrent.

For information about what you need to know about triangle centers and center functions, we give a short overview in Appendix A.

We make extensive use of areal coordinates (also known as barycentric coordinates) when analyzing points associated with triangles, such as the faces of a tetrahedron. For the reader not familiar with areal coordinates, we give the basics in Appendix B.

For points, lines, and planes in space, we make heavy use of tetrahedral coordinates. For the reader not familiar with tetrahedral coordinates, we present the needed information in Appendix C.

Throughout this paper, the notation $[XYZ]$ will denote the area of triangle XYZ .

2. COORDINATES FOR THE FACE CENTERS

When referring to an arbitrary tetrahedron (the reference tetrahedron), we will usually label the vertices A_1 , A_2 , A_3 , and A_4 . The lengths of the sides of the base ($\triangle A_1A_2A_3$) will be a_1 , a_2 , and a_3 , with edge a_i opposite vertex A_i . In the tetrahedron, the edge opposite the edge of length a_i will have length b_i . See Figure 1a.

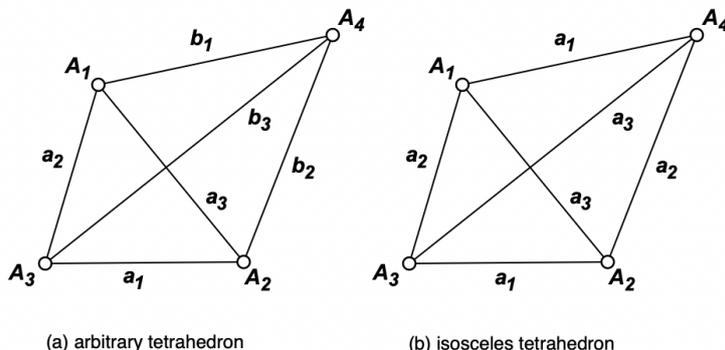


FIGURE 1. edge labeling

Thus, we have

$$A_2A_3 = a_1, \quad A_3A_1 = a_2, \quad A_1A_2 = a_3, \quad A_1A_4 = b_1, \quad A_2A_4 = b_2, \quad A_3A_4 = b_3.$$

If the tetrahedron has its opposite edges of equal length, then the tetrahedron is called an isosceles tetrahedron. See Figure 1b (not to scale). It is clear that in an isosceles tetrahedron, the four faces are congruent because they each have sides of length a_1 , a_2 , a_3 . In a sense, the isosceles tetrahedron “looks the same” from each vertex. This four-fold symmetry makes the isosceles tetrahedron be the figure in space that corresponds to the equilateral triangle in the plane. An equilateral triangle has 3 identical sides and an isosceles tetrahedron has 4 identical faces. The face of the tetrahedron opposite vertex A_i will be called face i of the tetrahedron. Its area will be denoted by F_i .

If (x_1, x_2, x_3) are the areal coordinates for a triangle center, then the tetrahedral coordinates for the corresponding center on face 4 of our reference tetrahedron $A_1A_2A_3A_4$ is $(x_1, x_2, x_3, 0)$. To see why this is true, consider the center P on face 4 (triangle $A_1A_2A_3$) of the tetrahedron. Then

$$[PA_1A_2A_4] : [PA_2A_3A_4] : [PA_3A_1A_4] = [PA_1A_2] : [PA_2A_3] : [PA_3A_1]$$

since these 4 tetrahedra have a common altitude from A_4 .

We will frequently have occasion to pick a point on each face of the tetrahedron. In such a case, the point on face i will be labelled P_i . It is often necessary to locate such a point based on its areal coordinates in face i . We must be careful how we set up the coordinate system on each face. Note that in an arbitrary tetrahedron, each face has the property that the labels associated with each edge (a_i or b_i) contains one label with subscript 1, one with subscript 2, and one with subscript 3. In order to maintain the 4-fold symmetry exhibited by an isosceles tetrahedron, under the mapping $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4$ we want the faces to transform as follows:

$$\triangle A_4A_3A_2 \rightarrow \triangle A_3A_4A_1 \rightarrow \triangle A_2A_1A_4 \rightarrow \triangle A_1A_2A_3.$$

Note that in each face, our labelling starts with the vertex opposite the edge whose label has subscript 1, then proceeds to the vertex opposite the edge whose label has subscript 2 and finally ends with the edge whose label has subscript 3. This induces the following correspondence between the edges:

$$\begin{aligned} (a_1, b_2, b_3, b_1, a_2, a_3) &\rightarrow (b_1, a_2, b_3, a_1, b_2, a_3) \\ &\rightarrow (b_1, b_2, a_3, a_1, a_2, b_3) \\ &\rightarrow (a_1, a_2, a_3, b_1, b_2, b_3). \end{aligned}$$

This mapping is shown in figure 2.

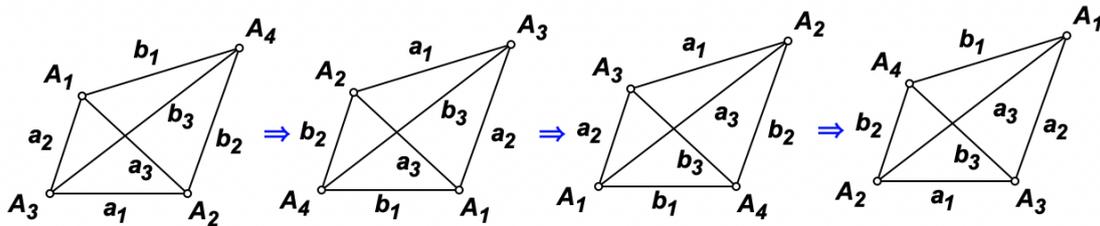


FIGURE 2. mapping

Let us now consider the mapping which takes A_1 into A_2 in this 4-fold symmetry. We start, by finding the tetrahedral coordinates for the point, P_1 , with areal coordinates (x_1, x_2, x_3) in face 1 of the reference tetrahedron. The first coordinate “ x_1 ” refers to the area formed by the point P_1 and the side of the triangle with a “1” as subscript. In this case, face 1 has sides of length a_1, b_2 , and b_3 , so the side we need is the side of length a_1 . On face 1, this side is opposite vertex A_4 of the reference tetrahedron and so the “ x_1 ” coordinate will appear as the 4th tetrahedral coordinate. Proceeding in this manner, we find that P_1 has tetrahedral coordinates $(0, x_3, x_2, x_1)$. This point wants to map to a point, P_2 , with the same areal coordinates in face 2. In face 1 ($\triangle A_4A_3A_2$), the coordinates correspond to areas associated with edges a_1, b_2 , and b_3 . In face 2 ($\triangle A_3A_4A_1$), the corresponding edges are b_1, a_2 , and b_3 . Point P_2 has tetrahedral coordinates $(x_3, 0, x_1, x_2)$ because

on face 2, edge b_1 is opposite vertex A_3 (so x_1 moves to the 3rd coordinate in the tetrahedral system), a_2 is opposite vertex A_4 (so x_2 moves to the 4th coordinate in the tetrahedral system), and b_3 is opposite vertex A_1 (so x_3 moves to the 1st coordinate in the tetrahedral system).

In other words, given a point in the plane with areal coordinates (x_1, x_2, x_3) , the corresponding points in the faces of our reference tetrahedron are:

$$\begin{aligned} \text{Face 1 :} & \quad (0, x_3, x_2, x_1) \\ \text{Face 2 :} & \quad (x_3, 0, x_1, x_2) \\ \text{Face 3 :} & \quad (x_2, x_1, 0, x_3) \\ \text{Face 4 :} & \quad (x_1, x_2, x_3, 0) \end{aligned}$$

where we have associated face 4 with the original plane triangle.

If the original point is a center, with areal coordinates

$$(f(a_1, a_2, a_3), f(a_2, a_3, a_1), f(a_3, a_1, a_2)),$$

then the corresponding points on the faces of the tetrahedron are:

$$\begin{aligned} \text{Face 1 :} & \quad (0, f(b_3, b_2, a_1), f(b_2, a_1, b_3), f(a_1, b_3, b_2)) \\ \text{Face 2 :} & \quad (f(b_3, b_1, a_2), 0, f(b_1, a_2, b_3), f(a_2, b_3, b_1)) \\ \text{Face 3 :} & \quad (f(b_2, b_1, a_3), f(b_1, a_3, b_2), 0, f(a_3, b_2, b_1)) \\ \text{Face 4 :} & \quad (f(a_1, a_2, a_3), f(a_2, a_3, a_1), f(a_3, a_1, a_2), 0). \end{aligned}$$

Kimberling [11] and [12] has collected the trilinear coordinates for over 40,000 centers associated with a triangle. He lists the trilinear coordinates in terms of the sides a, b, c of the reference triangle and trigonometric functions of A, B, C , the angles of the reference triangle. Only the first coordinate is given, for if this coordinate is $f(a, b, c, A, B, C)$, then the other coordinates are $f(b, c, a, B, C, A)$ and $f(c, a, b, C, A, B)$ respectively.

We wish to study points associated with a tetrahedron based on the lengths of the 6 edges of the tetrahedron. The six edge lengths are independent quantities. Involving other quantities such as the face areas or trigonometric functions of the face or dihedral angles would yield expressions containing dependent variables and would complicate the process of determining if such expressions are identically 0 for all tetrahedra. We thus need to remove the presence of angles from Kimberling's data. Since all the trigonometric functions present can be expressed in terms of sine's and cosine's of the angles of the reference triangle, the following formulas suffice to remove all reference to these angles:

$$\begin{aligned} \sin A &= \frac{2K}{bc} \\ \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \end{aligned}$$

where K denotes the area of the reference triangle. The first formula comes from the well-known area formula: $K = \frac{1}{2}bc \sin A$; and the second formula is The Law of Cosines. Similar expressions hold for angles B and C .

Factors (such as K or $a + b + c$) that would be common to all three coordinates are then removed.

The presence of a K in the denominator of any fraction involved is cumbersome and was removed by replacing terms of the form $x/(y + zK)$ by $x(y - zK)/(y^2 - z^2K^2)$. This leaves all square roots in the numerators.

The variable K is then replaced by its equivalent expression in terms of the sides of the triangle (Heron's Formula), namely

$$K = \frac{1}{4} \sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}.$$

Since tetrahedral coordinates are barycentric, if (x, y, z, w) are the coordinates for one of the above centers in our reference tetrahedron, then the tetrahedral coordinates for the corresponding center in a tetrahedron with vertices P_1, P_2, P_3 , and P_4 are $xP_1 + yP_2 + zP_3 + wP_4$, where xP_1 denotes the scalar product of x and the vector P_1 , etc. Algorithmically, the desired coordinates are the dot product of the vectors (x, y, z, w) and (P_1, P_2, P_3, P_4) .

3. TYPES OF TETRAHEDRA

The types of tetrahedra investigated are listed in the following table.

Types of Tetrahedra Considered		
Tetrahedron Type	Geometric Definition	Algebraic Condition $i = 1, 2, 3$
General	no restrictions placed on the edges	none
Isosceles	faces are congruent	$a_i = b_i$
Circumscribable	edges are tangent to a sphere	$a_i + b_i = \text{constant}$
Isodynamic	symmedians are concurrent	$a_i b_i = \text{constant}$
Orthocentric	opposite edges are perpendicular	$a_i^2 + b_i^2 = \text{constant}$
Harmonic	n/a	$1/a_i + 1/b_i = \text{constant}$

Only tetrahedra that have the requisite 4-fold symmetry were studied. Thus, for example, trirectangular tetrahedra are not included in this study. We would have liked to have investigated isogonic tetrahedra (ones in which the cevians to the points of tangency of the insphere are concurrent, ([2, p. 328])), but the corresponding algebraic condition was too messy to be manageable. The concept of a harmonic tetrahedron was invented for this study and has a few interesting properties, but perhaps not enough to warrant future study. The other types of tetrahedra are well known and information about them can be found in [2].

4. METHODOLOGY

For this study, we considered the first 101 triangle centers listed in [11], X1 through X101, as well as a few other centers, listed in the following table.

Triangle Centers Considered	
Triangle Center	Trilinears
X1–X101	see [11]
Y1	$1/(a^2(b+c) - abc)$
Y2	$a/((b-a)(c-a))$
Y3	$a^2(b+c)$
Y4	$1/(a^2(b+c))$
Y5	$a/(b^2+c^2)$
Y6	$a^2(b^2+c^2)$
Y7	$1/(a^2(b^2+c^2))$
Y8	$(b^2+c^2)/a$
Y9	$a(b+c-2a)$
Y10	$1/(a(b+c-2a))$
Y11	$a/(b+c-2a)$
Y12	$a^2(b+c-2a)$
Y13	$1/(a^2(b+c-2a))$
Y14	$b+c-bc/a$
r-power point	a^r
Z1	$a^r(b+c)$
Z2	$a^r(b^2+c^2)$
Z3	$a^r(b+c-a)$
Z4	$a^r(b+c-2a)$
Z5	$a^r(b^2+c^2-a^2)$
Z6	$a^r(b^3+c^3)$
Z7	$a^r(b^2+c^2+bc)$
Z8	$2a^r+b^r+c^r$
Z9	$(b^r+c^r)/a$
Z10	$(b^r+c^r-a^r)/a$
Z11	$(b^r+c^r+2a^r)/a$
power center	$a^r g[b, c]$
arbitrary center	$f[a, g[b, c]]$
areal center	$f[a, b, c]/a$

For each type of tetrahedron considered, and for each triangle center considered, we computed the tetrahedral coordinates of these centers on each face of the tetrahedron.

Once we had located these four centers, we then used Mathematica to run a barrage of tests on these four points to see if they satisfied any special properties. Since these tests involved algebraic coordinates (i.e. we were not looking at specific tetrahedra with numerical sides), any results found constitute a proof that the result is true and not merely a conjecture based on numerical evidence. These results are stated in sections 5 through 10. The proofs are by coordinate geometry, mechanically performed by the Mathematica program which was written to compute all the necessary lengths and coordinates and then confirm the claimed results symbolically.

Most of these results are new, however, some of them may have previously appeared in the literature. We give references, when known. A few related results appeared as problem 3 in the 15th Summer Conference of the International Mathematical Tournament of Towns, [15].

First a few definitions.

Definition 1. The original tetrahedron is known as the *reference tetrahedron*.

Definition 2. The tetrahedron formed by the four centers is called the *central tetrahedron*.

Definition 3. The line segment from a vertex of the reference tetrahedron to the center on the opposite face is called a *cevian*.

Definition 4. Four skew lines in space are said to form a *hyperbolic group* if there is an infinite number of lines that meet all four of these lines.

According to Altshiller-Court ([2, p. 10]), “Such a group is often the space analog of three concurrent lines in the plane.”

Definition 5. The four skew lines are part of an infinite family of lines that form a ruled surface known as a *hyperboloid* of one sheet.

Definition 6. By a *space center* of a tetrahedron, we mean one of: centroid, circumcenter, incenter, Monge point, or Euler point. These are described in the following table.

Tetrahedron Centers Considered	
Space Center	Description
centroid	intersection point of medians
circumcenter	center of circumscribed sphere
incenter	center of inscribed sphere
Monge point	symmetric of circumcenter with respect to centroid
Euler point	center of 12-point sphere

More background information about these centers is given in Appendix E.

The properties that were checked for are listed in the table below.

Properties Considered	
Property 1	The cevians to the four centers are concurrent.
Property 2	The cevians to the four centers form a hyperbolic group.
Property 3	The four centers are coplanar.
Property 4	The four centers are collinear.
Property 5	The normals to the faces at the centers concur.
Property 6	The faces of the central tetrahedron are parallel to the faces of the reference tetrahedron.
Property 7	The central tetrahedron is isosceles.
Property 8	The central tetrahedron is regular.
Property 9	The central tetrahedron is isodynamic.
Property 10	The central tetrahedron is circumscribable.
Property 11	The central tetrahedron is orthocentric.
Property 12	The central tetrahedron is similar to the reference tetrahedron.
Property 13	The cevians to the four centers have the same length.
Property 14	The central tetrahedron has a space center in common with some space center of the reference tetrahedron.
Property 15	The central tetrahedron has a space center on the Euler line of the reference tetrahedron.
Property 16	The reference tetrahedron has a space center on the Euler line of the central tetrahedron.

If the cevians concurred, we also checked to see if the point of concurrence was a space center of the reference tetrahedron or if it lied on the Euler line of the reference tetrahedron. Also, if the four cevians formed a hyperbolic group, we computed the center of the hyperboloid for which these cevians were generators and checked this point to see if it was a space center of the reference tetrahedron (or on its Euler line).

To find the center of the hyperboloid, we used the following result:

Proposition 4.1 ([13]). *If L_1 , L_2 , and L_3 are three lines that determine a hyperboloid of one sheet, then if one draws planes through each of these lines parallel to the two others, then we get a parallelepiped. The center of this parallelepiped is the center of the hyperboloid.*

Thus, our test was as follows: Use formula 17 to find the plane through L_1 and parallel to L_2 . (See Appendix D for formulas using tetrahedral coordinates.) Let X be the point of intersection of L_3 with this plane (found via formula 18). Similarly, find the plane through L_1 parallel to L_3 . Let Y be the intersection of this plane and L_2 . Then the center of the hyperboloid is the midpoint of segment XY .

To make some of the computation of properties 1-16 easier, we first checked the property for a specific tetrahedron with numerical sides. If the property was false (using exact arithmetic) for this numerical case, then we did not bother checking to see if the property was algebraically true in general.

5. RESULTS FOUND FOR ARBITRARY TETRAHEDRA

The following results were discovered and proven by our computer program.

Theorem 5.1. *Consider the centroids on each face of an arbitrary tetrahedron. Then*

- (a) *The faces of the central tetrahedron are parallel to the corresponding faces of the reference tetrahedron.*
- (b) *The cevians to the centroids concur at the centroid of the reference tetrahedron.*
- (c) *The central tetrahedron is similar to the reference tetrahedron.*
- (d) *The central centroid coincides with the reference centroid.*
- (e) *The central circumcenter coincides with the reference Euler point.*
- (f) *The central Monge point lies on the reference Euler line (at $2/3$).*
- (g) *The central Euler point lies on the reference Euler line (at $8/9$).*
- (h) *The reference circumcenter lies on the central Euler line (at 4).*
- (i) *The reference Monge point lies on the central Euler line (at -2).*

Theorem 5.2. *For an arbitrary tetrahedron, the normals at the circumcenters of each face concur at the circumcenter of the reference tetrahedron.*

Theorem 5.3. *For an arbitrary tetrahedron, the lines to the r -power points form a hyperbolic group. These include the incenters, the centroids, and the symmedian points.*

Note. Results found for specific tetrahedra that are immediate consequences of results in this section for arbitrary tetrahedra will not necessarily be listed again below.

Theorem 5.4. *For fixed r , the $2a^r + b^r + c^r$ points of an arbitrary tetrahedron form a central tetrahedron that has the same centroid as the reference tetrahedron.*

6. RESULTS FOUND FOR ISOSCELES TETRAHEDRA

The following results were discovered and proven by our computer program.

Theorem 6.1. *Consider an arbitrary center on each face of an isosceles tetrahedron. (The same type of center is considered on each face.) Then*

- (a) *The cevians to these centers have the same length.*
- (b) *The central tetrahedron is isosceles.*
- (c) *The central tetrahedron has the same centroid as the reference tetrahedron.*
- (d) *The cevians form a hyperbolic group.*

7. RESULTS FOUND FOR CIRCUMSCRIPTIBLE TETRAHEDRA

The following results were discovered and proven by our computer program.

Theorem 7.1. *For a circumscribable tetrahedron (in which $a_i + b_i = t$, $i = 1, 2, 3$),*

- (a) *The cevians to the Gergonne points concur.*
The 4th coordinate of the intersection point is
 $(a_2 + a_3 - a_1)(a_3 + a_1 - a_2)(a_1 + a_2 - a_3)$.
- (b) *The cevians to the Nagel points concur.*
The 4th coordinate of the intersection point is $a_1 + a_2 + a_3 - 2t$.
This equals $\frac{S}{2} - S_4$ where S_i is the sum of the edges at A_i and $S = \sum S_i$.
- (c) *The Feuerbach points are coplanar.*
- (d) *The normals at the incenters concur.*
- (e) *The normals at the X_{40} points concur.*
Note that X_{40} is collinear with the incenter and circumcenter.

Theorem 7.2. *For a circumscribable tetrahedron, the lines to the following triangle centers form a hyperbolic group:*

- (a) *Gergonne points (and their inverses)*
- (b) *Nagel points (and their inverses)*
- (c) *Mittelpunkte (and their inverses)*
- (d) *X_{41} points (and their inverses)*
- (e) *Feuerbach points (and their inverses).*

8. RESULTS FOUND FOR ISODYNAMIC TETRAHEDRA

The following results were discovered and proven by our computer program.

Theorem 8.1. *For an isodynamic tetrahedron,*

- (a) *The cevians to any power point concur.*
The 4th coordinate of the intersection point is $a_1 a_2^{r+1} a_3$.
- (b) *The Feuerbach points are coplanar.*
- (c) *The X_{44} points are coplanar.*
- (d) *The Lemoine axes are coplanar.*
- (e) *The circumcenter of the X_{76} points (3rd power point inverses) coincides with the reference centroid.*

Theorem 8.2. *For an isodynamic tetrahedron, the lines to the following triangle centers form a hyperbolic group:*

- (a) *Spieker centers (and their inverses)*
- (b) *X37 points (and their inverses)*
- (c) *X38 points (and their inverses)*
- (d) *Brocard midpoint (and their inverses)*
- (e) *X42 points (and their inverses)*
- (f) *X106 points*
- (g) *X107 points*
- (h) *X108 points*
- (i) *X109 points*
- (j) *X110 points*
- (k) *X111 points*

9. RESULTS FOUND FOR ORTHOCENTRIC TETRAHEDRA

The following results were discovered and proven by our computer program.

Theorem 9.1. *For an orthocentric tetrahedron (in which $a_i^2 + b_i^2 = t$, $i = 1, 2, 3$),*

- (a) *The cevians to the orthocenters concur.*
The 4th coordinate of the intersection point is
 $(a_2^2 + a_3^2 - a_1^2)(a_3^2 + a_1^2 - a_2^2)(a_1^2 + a_2^2 - a_3^2)$.
- (b) *The cevians to the isotomic conjugates of the orthocenters concur.*
The 4th coordinate of the intersection point is $a_1^2 + a_2^2 + a_3^2 - 2t$. This equals
 $\frac{T}{2} - T_4$ *where T_i is the sum of the squares of the edges at A_i and $T = \sum T_i$.*
- (c) *The centroid of the 9-point centers coincides with the reference centroid.*
- (d) *The centroid of the orthocenters coincides with the reference Monge point.*
- (e) *The circumcenter of the orthocenters lies on the reference Euler line.*
- (f) *The Monge point of the orthocenters lies on the reference Euler line.*
- (g) *The centroid of the X53 points coincide with the reference Monge point.*

Theorem 9.2. *For an orthocentric tetrahedron,*

- (a) *The normals at the circumcenters concur.*
- (b) *The normals at the centroids concur.*
- (c) *The normals at the orthocenters concur.*
- (d) *The normals at the nine point centers concur.*
- (e) *The normals at the De Longchamps points concur.*

Note that these five centers lie on the Euler line and have constant ratio distances apart.

Theorem 9.3. *For an orthocentric tetrahedron, the lines to the following triangle centers form a hyperbolic group:*

- (a) *circumcenters*
- (b) *Crucial points (and their inverses)*
- (c) *X25 points*
- (d) *X48 points (and their inverses)*

10. RESULTS FOUND FOR HARMONIC TETRAHEDRA

The following results were discovered and proven by our computer program.

Theorem 10.1. *For a harmonic tetrahedron,*

- (a) *The Feuerbach points are coplanar.*
- (b) *The cevians to the X117 points (and their isotomic conjugates, the X102 points) concur.*

Theorem 10.2. *For a harmonic tetrahedron, the lines to the following triangle centers form a hyperbolic group:*

- (a) *X43 points (and their inverses)*
- (b) *X102 points*
- (c) *X117 points*

Note that $X43 = 1/b + 1/c - 1/a$, $X102 = a(1/b + 1/c - 1/a)$, and $X117 = (1/b + 1/c - 1/a)/a$.

11. GENERAL RESULTS ABOUT CONCURRENT CEVIANS

The data collected by our program suggested (but did not prove) the following results. Thus, independent proofs are needed.

Lemma 11.1. *Let $P_1 = (0, y_1, z_1, w_1)$ and $P_2 = (x_2, 0, z_2, w_2)$ be two points on faces 1 and 2 of the reference tetrahedron. Then the condition that the lines A_iP_i , $i = 1, 2$ intersect (or be parallel) is*

$$\frac{z_1}{w_1} = \frac{z_2}{w_2}.$$

Proof. From formulas 7 and 11, the condition is

$$\begin{vmatrix} 0 & y_1 & z_1 & w_1 \\ x_2 & 0 & z_2 & w_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 0.$$

which reduces to the formula claimed. □

Geometric interpretation. Let P_1 and P_2 be points on the faces opposite vertices A_1 and A_2 , respectively, of tetrahedron $A_1A_2A_3A_4$. Then lines A_1P_1 and A_2P_2 intersect if and only if

$$[P_1A_2A_3][P_2A_4A_1] = [P_1A_2A_4][P_2A_1A_3].$$

Here “intersect” also includes being parallel.

Figure 3 shows a top view of tetrahedron $A_1A_2A_3A_4$ with a point taken on faces 1 and 2. The condition is that the product of the areas of the yellow triangles equals the product of the areas of the green triangles.

Lemma 11.2. *If A_iP_i , $i = 1, 2, 3$, meet in pairs, then all three lines meet at a point.*

Proof. Let the lines meet in pair at points Q_1, Q_2, Q_3 . Then the plane through Q_1, Q_2 , and Q_3 contains the three lines and hence the three vertices A_1, A_2, A_3 . Thus A_1, A_2, A_3 would lie on a plane other than the base plane, a contradiction. □

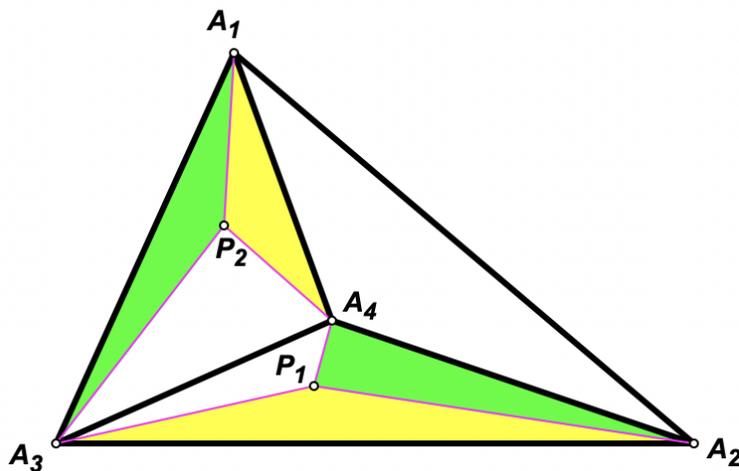


FIGURE 3. A_1P_1 intersects A_2P_2 if product of yellow areas equals product of green areas

Corollary 11.3. Let $P_1 = (0, y_1, z_1, w_1)$, $P_2 = (x_2, 0, z_2, w_2)$, $P_3 = (x_3, y_3, 0, w_3)$ be three points on faces 1, 2, and 3 of the reference tetrahedron. Then the condition that the lines A_iP_i , $i = 1, 2, 3$ concur (or be parallel) is

$$\frac{z_1}{w_1} = \frac{z_2}{w_2}, \quad \frac{y_1}{w_1} = \frac{y_3}{w_3}, \quad \frac{x_2}{w_2} = \frac{x_3}{w_3}.$$

Corollary 11.4 (The Concurrence Condition). The condition for the concurrence of cevians to two centers (each with center function $F(a, b, c)$) on faces 1 and 2 of the reference tetrahedron is

$$F(b_2, a_1, b_3)F(a_2, b_3, b_1) = F(a_1, b_3, b_2)F(b_1, a_2, b_3).$$

Theorem 11.5. If P_1 and P_2 are points on faces 1 and 2 of our reference tetrahedron such that the cevians to P_1 and P_2 meet, then the cevians to the isotomic conjugates of P_1 and P_2 meet.

Proof. In areal coordinates, the isotomic conjugate of (x, y, z) is $(1/x, 1/y, 1/z)$. The concurrence condition therefore becomes

$$\frac{1}{F(b_2, a_1, b_3)} \cdot \frac{1}{F(a_2, b_3, b_1)} = \frac{1}{F(a_1, b_3, b_2)} \cdot \frac{1}{F(b_1, a_2, b_3)}$$

which is equivalent to the original condition. \square

Corollary 11.6 ([2, p. 139]). If the four cevians to corresponding face centers concur, then the four cevians to the isotomic conjugates of these centers also concur.

Corollary 11.7. If cevians to the triangle centers with center function $F(a, b, c)$ concur, then so do cevians to the centers with center function $F(a, b, c)^r$ for any r .

Theorem 11.8. The centroid is the only triangle center with the property that in any isosceles tetrahedron, the cevians to these face centers concur.

Proof. Suppose $F(a, b, c) = af(a, b, c)$ is such a center function. The algebraic condition for this to be true is obtained by substituting $b_i = a_i$ in the concurrence

condition to get $F(a_1, a_2, a_3)^2 = F(a_2, a_1, a_3)^2$. This implies that

$$(1) \quad F(a_1, a_2, a_3) = -F(a_2, a_1, a_3)$$

or

$$(2) \quad F(a_1, a_2, a_3) = F(a_2, a_1, a_3)$$

for all a_1, a_2 , and a_3 .

If condition (1) holds, then we would have

$$F(a_1, a_2, a_3) = -F(a_2, a_1, a_3)$$

$$F(a_2, a_3, a_1) = -F(a_3, a_2, a_1)$$

$$F(a_3, a_1, a_2) = -F(a_1, a_3, a_2)$$

since the equality must be true for all values of its arguments. Since $F(a, b, c) = F(a, c, b)$, multiplying these three equations together yields $1 = -1$, a contradiction.

If condition (2) holds, then we would have

$$\frac{F(a_1, a_2, a_3)}{F(a_2, a_3, a_1)} = 1$$

or

$$F(a_1, a_2, a_3) : F(a_2, a_3, a_1) : F(a_3, a_1, a_2) = 1 : 1 : 1$$

so that F represents the centroid. □

Corollary 11.9. *The centroid is the only triangle center with the property that in any tetrahedron, the cevians to these face centers concur.*

Proof. Since the cevians concur for *any* tetrahedron, they must surely concur for any isosceles tetrahedron. But the previous theorem rules this possibility out. □

The following lemma is well known:

Lemma 11.10 (Power Lemma). *If $f(x)$ is a nonzero function satisfying*

$$f(xy) = f(x)f(y)$$

for all x and y , then $f(x) = x^r$ for some constant r .

Theorem 11.11. *The power points are the only triangle centers with the property that in any isodynamic tetrahedron, the cevians to these face centers concur.*

Proof. The concurrence condition becomes

$$F\left(\frac{t}{a_2}, a_1, \frac{t}{a_3}\right)F\left(a_2, \frac{t}{a_3}, \frac{t}{a_1}\right) = F\left(a_1, \frac{t}{a_3}, \frac{t}{a_2}\right)F\left(\frac{t}{a_1}, a_2, \frac{t}{a_3}\right)$$

for all a_1, a_2, a_3 , and t . We can write this as

$$\frac{F\left(\frac{t}{a_2}, a_1, \frac{t}{a_3}\right)}{F\left(a_1, \frac{t}{a_2}, \frac{t}{a_3}\right)} = \frac{F\left(\frac{t}{a_1}, a_2, \frac{t}{a_3}\right)}{F\left(a_2, \frac{t}{a_1}, \frac{t}{a_3}\right)}.$$

Since this is true for all a_2 , it will be true if we replace a_2 by t/a_2 to get

$$\frac{F\left(a_2, a_1, \frac{t}{a_3}\right)}{F\left(a_1, a_2, \frac{t}{a_3}\right)} = \frac{F\left(\frac{t}{a_1}, \frac{t}{a_2}, \frac{t}{a_3}\right)}{F\left(\frac{t}{a_2}, \frac{t}{a_1}, \frac{t}{a_3}\right)}.$$

Since F is homogeneous, we have

$$\frac{F(a_2, a_1, \frac{t}{a_3})}{F(a_1, a_2, \frac{t}{a_3})} = \frac{F(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3})}{F(\frac{1}{a_2}, \frac{1}{a_1}, \frac{1}{a_3})}.$$

Let $t = a_3x$ to get

$$\frac{F(a_2, a_1, x)}{F(a_1, a_2, x)} = \frac{F(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3})}{F(\frac{1}{a_2}, \frac{1}{a_1}, \frac{1}{a_3})}.$$

The right-hand side is independent of x , and so we can define

$$(3) \quad G(a_1, a_2) = \frac{F(a_2, a_1, x)}{F(a_1, a_2, x)}.$$

Thus

$$F(a, b, c) = G(a, b)F(b, a, c) = G(a, b)G(b, a)F(a, b, c)$$

and so

$$(4) \quad G(a, b)G(b, a) = 1$$

for all a and b . Similarly,

$$\begin{aligned} F(a, b, c) &= G(a, b)F(b, a, c) = G(a, b)F(b, c, a) \\ &= G(a, b)G(b, c)F(c, b, a) = G(a, b)G(b, c)F(c, a, b) \\ &= G(a, b)G(b, c)G(c, a)F(a, c, b) = G(a, b)G(b, c)G(c, a)F(a, b, c) \end{aligned}$$

and so

$$(5) \quad G(a, b)G(b, c)G(c, a) = 1$$

for all a, b and c . Using the homogeneity of F in equation (3), we can divide all arguments by a_2 to get

$$G(a_1/a_2, 1) = \frac{F(1, a_1/a_2, x/a_2)}{F(a_1/a_2, 1, x/a_2)} = \frac{F(a_2, a_1, x)}{F(a_1, a_2, x)} = G(a_1, a_2).$$

Thus, if we define

$$g(x) = G(x, 1),$$

then we get analogs of equations (4) and (5):

$$g\left(\frac{a}{b}\right)g\left(\frac{b}{a}\right) = 1$$

and

$$g\left(\frac{a}{b}\right)g\left(\frac{b}{c}\right)g\left(\frac{c}{a}\right) = 1.$$

Now

$$g\left(\frac{a}{b}\right)g\left(\frac{b}{c}\right) = 1/g\left(\frac{c}{a}\right) = g\left(\frac{a}{c}\right).$$

Let $x = a/b$ and $y = b/c$ to get

$$g(x)g(y) = g(xy)$$

for all x and y . By the Power Lemma, we must have $g(x) = x^r$ for some r . Hence

$$\begin{aligned} F(a, b, c) : F(b, c, a) : F(c, a, b) &= \frac{F(a, b, c)}{F(a, b, c)} : \frac{F(b, c, a)}{F(a, b, c)} : \frac{F(c, a, b)}{F(a, b, c)} \\ &= 1 : G(b, a) : G(c, a) \\ &= 1 : g\left(\frac{b}{a}\right) : g\left(\frac{c}{a}\right) \\ &= 1 : \left(\frac{b}{a}\right)^r : \left(\frac{c}{a}\right)^r \\ &= a^r : b^r : c^r \end{aligned}$$

and thus the center is a power point. \square

Theorem 11.12. *Suppose the edges of a tetrahedron satisfy the condition $h(a_i) + h(b_i) = t$, $i = 1, 2, 3$, for some function $h(x)$ and some constant, t . Then the cevians to the face centers with (areal) center function $[h(b) + h(c) - h(a)]^r$ concur for any r .*

Proof. The concurrence condition is

$$\begin{aligned} [h(a_1) + h(b_3) - h(b_2)]^r [h(b_3) + h(b_1) - h(a_2)]^r \\ = [h(b_3) + h(b_2) - h(a_1)]^r [h(a_2) + h(b_3) - h(b_1)]^r. \end{aligned}$$

This is equivalent to

$$\begin{aligned} [h(a_1) + h(b_3) + h(a_2) - t]^r [h(b_3) + h(b_1) + h(b_2) - t]^r \\ = [h(b_3) + h(b_2) + h(b_1) - t]^r [h(a_2) + h(b_3) + h(a_1) - t]^r \end{aligned}$$

which is easily seen to be an identity. \square

Corollary 11.13. *In a circumscribable tetrahedron, the cevians to the face centers with areal center function $(b + c - a)^r$ concur. This includes the Gergonne point and its isotomic conjugate, the Nagel point.*

Corollary 11.14. *In an orthocentric tetrahedron, the cevians to the face centers with areal center function $(b^2 + c^2 - a^2)^r$ concur. This includes the orthocenter and its isotomic conjugate.*

Corollary 11.15. *In a harmonic tetrahedron, the cevians to the face centers with areal center function $(1/b + 1/c - 1/a)^r$ concur. This includes the X117 point and its isotomic conjugate, the X102 point.*

Corollary 11.16. *In an isodynamic tetrahedron, the cevians to the power points concur. This includes the incenter, centroid, symmedian point and their isogonal and isotomic conjugates.*

Proof. Take $h(x) = \log x$ in the previous theorem. \square

Conjecture 11.17. *Suppose the edges of a tetrahedron satisfy the condition $h(a_i) + h(b_i) = t$, $i = 1, 2, 3$, for some power function $h(x) = x^n$ and some constant, t . If cevians to four corresponding face centers are concurrent, then the center function for these face centers must be of the form $[h(b) + h(c) - h(a)]^r$ for some r .*

Theorem 11.18. *If cevians to the points $h(b) + h(c) - h(a)$ concur, then the edges of the tetrahedron satisfy $h(a_i) + h(b_i) = t$, for $i = 1, 2, 3$.*

Proof. The concurrency condition (for centers on faces 1 and 2) becomes

$$\begin{aligned} & [h(a_1) + h(b_3) - h(b_2)][h(b_3) + h(b_1) - h(a_2)] \\ &= [h(b_3) + h(b_2) - h(a_1)][h(a_2) + h(b_3) - h(b_1)]. \end{aligned}$$

Simple algebra transforms this into the equation

$$h(b_3)[h(a_1) + h(b_1)] = h(b_3)[h(a_2) + h(b_2)]$$

from which we conclude that

$$h(a_1) + h(b_1) = h(a_2) + h(b_2).$$

By symmetry, analogous results are true for any two faces of the reference tetrahedron, so $h(a_i) + h(b_i)$ is constant for all i . \square

Corollary 11.19 ([2, p. 299]). *If cevians to the Nagel points concur, then the tetrahedron is isodynamic.*

Corollary 11.20 ([2, p. 299]). *If cevians to the Gergonne points concur, then the tetrahedron is isodynamic.*

Corollary 11.21. *If cevians to the orthocenters concur, then the tetrahedron is orthocentric.*

Corollary 11.22. *If cevians to the $(1/b + 1/c - 1/a)$ centers concur, then the tetrahedron is harmonic.*

Corollary 11.23. *For a fixed $r \neq 0$, if cevians to the r -power points concur, then the tetrahedron is isodynamic.*

This generalizes proposition 841 of [2] (which was for the Symmedian point only).

Proof. The a^r centers are the same as the $a^r b^r c^r / a^r$ centers, so the result follows by taking $h(x) = \log x$. \square

12. GENERAL RESULTS ABOUT HYPERBOLIC LINES

Theorem 12.1. *Let $P_1 = (0, y_1, z_1, w_1)$, $P_2 = (x_2, 0, z_2, w_2)$, and $P_3 = (x_3, y_3, 0, w_3)$ be three points on faces 1, 2, and 3 of the reference tetrahedron. Then the condition that there is a line through vertex A_4 that meets all three of the lines $A_i P_i$, $i = 1, 2, 3$ is*

$$z_1 x_2 y_3 = y_1 z_2 x_3.$$

This line is called a spear line. The spear line meets face $A_1 A_2 A_3$ at the point $(x_3 y_1, y_1 y_3, y_3 z_1, 0)$. (This point is known as the spear trace.)

Proof. This result was found by computer but could easily be carried out by hand. Formula 16 gives us the equation of the plane, E , through A_4 and $A_3 P_3$. Any spear line must clearly lie in this plane. Formula 18 determines the point, Q_1 , where line $A_1 P_1$ meets plane E . Then $A_4 Q_1$ must be the desired spear line. Similarly, we can find the point Q_2 , where line $A_2 P_2$ meets plane E . Then $A_4 Q_2$ must also be the desired spear line. Thus the condition is that points A_4 , Q_1 , and Q_2 colline. Formula 3 gives us this condition. Upon simplifying the result, the computer came up with $z_1 x_2 y_3 = y_1 z_2 x_3$ as the algebraic condition. We can then find the intersection of the common line $A_4 Q_1 Q_2$ and the plane $A_1 A_2 A_3$ to get the coordinates of the spear trace. \square

Geometric interpretation.

Let P_1 , P_2 , and P_3 be points on the faces opposite vertices A_1 , A_2 , and A_3 , respectively, of tetrahedron $A_1A_2A_3A_4$. Then there is a line through A_4 that meets lines A_1P_1 , A_2P_2 and A_3P_3 if and only if

$$[P_1A_2A_4][P_2A_3A_4][P_3A_1A_4] = [P_1A_3A_4][P_2A_1A_4][P_3A_2A_4]$$

where $[XYZ]$ denotes the area of triangle XYZ .

Figure 4a shows a top view of tetrahedron $A_1A_2A_3A_4$. Figure 4b then shows a point taken on faces 1, 2, and 3. The condition is that the product of the areas of the yellow triangles equals the product of the areas of the green triangles.

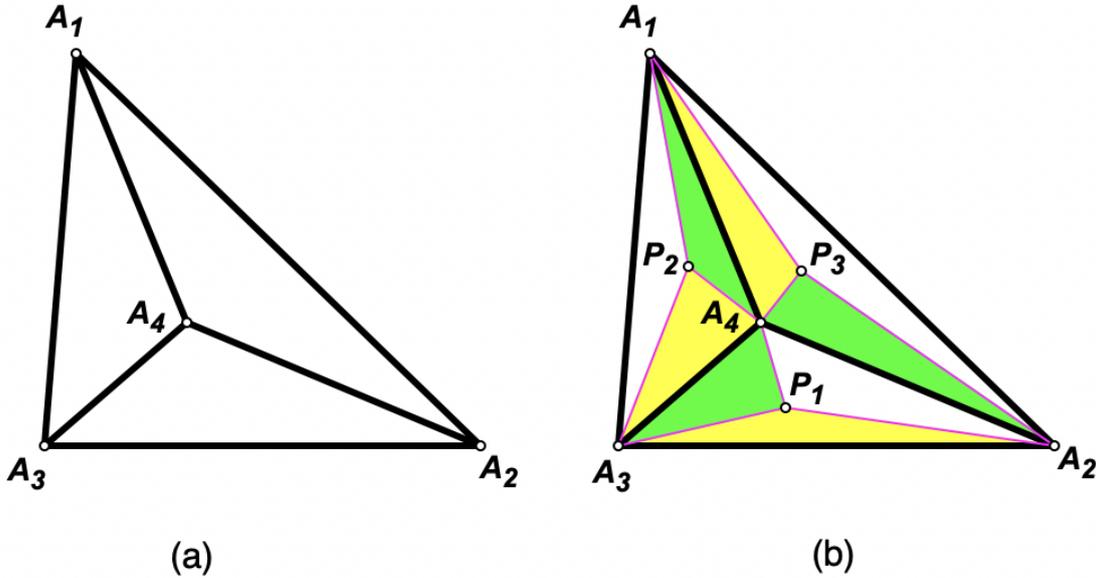


FIGURE 4. There is a line through A_4 that meets A_iP_i , $i = 1, 2, 3$ if and only if product of yellow areas equals product of green areas.

We say that a center function $F(a, b, c) = af(a, b, c)$ is a hyperbolic center function if the cevians from the vertices of a tetrahedron to these centers on the opposite faces form a hyperbolic group.

Proposition 12.2 ([2, p. 11]). *If four given mutually skew lines passing through the four vertices of a tetrahedron are such that through each vertex it is possible to draw a spear line meeting the three lines passing through the remaining three vertices, then the four given lines form a hyperbolic group.*

Corollary 12.3 (Hyperbolic Condition). *The condition that $F(a, b, c) = af(a, b, c)$ be a hyperbolic center function is:*

$$F(b_2, a_1, b_3)F(b_3, b_1, a_2)F(b_1, a_3, b_2) = F(b_3, b_2, a_1)F(b_1, a_2, b_3)F(b_2, b_1, a_3).$$

This follows from the coordinates for the corresponding center on each face found in section 2 and the preceding Proposition. Note that symmetry conditions imply that we need only find the condition that one spear line exists (instead of all 4).

Corollary 12.4. *In an isosceles tetrahedron, all center functions are hyperbolic.*

Proof. When $b_i = a_i$, the hyperbolic condition becomes

$$F(a_2, a_1, a_3)F(a_3, a_1, a_2)F(a_1, a_3, a_2) = F(a_3, a_2, a_1)F(a_1, a_2, a_3)F(a_2, a_1, a_3)$$

which is clearly an identity since a center function $F(a, b, c)$ is symmetric in b and c . \square

Theorem 12.5. *If $F(a, b, c)$ is a hyperbolic center function, then so is $a^r F(a, b, c)^a$.*

Proof. The hyperbolic condition becomes

$$\begin{aligned} b_2^r F(b_2, a_1, b_3)^a b_3^r F(b_3, b_1, a_2)^a b_1^r F(b_1, a_3, b_2)^a \\ = b_3^r F(b_3, b_2, a_1)^a b_1^r F(b_1, a_2, b_3)^a b_2^r F(b_2, b_1, a_3)^a \end{aligned}$$

which is an immediate consequence of the original condition. \square

Theorem 12.6. *If cevians to corresponding centers on each face of a tetrahedron form a hyperbolic group, then so do cevians to the isotomic conjugates of those centers.*

Proof. Since in areal coordinates, the isotomic conjugate of a center (x, y, z) is $(1/x, 1/y, 1/z)$, the hyperbolic condition becomes

$$\frac{1}{F(b_2, a_1, b_3)} \frac{1}{F(b_3, b_1, a_2)} \frac{1}{F(b_1, a_3, b_2)} = \frac{1}{F(b_3, b_2, a_1)} \frac{1}{F(b_1, a_2, b_3)} \frac{1}{F(b_2, b_1, a_3)}$$

which is equivalent to the original condition. \square

Corollary 12.7 ([2, p. 332]). *If cevians to corresponding centers on each face of a tetrahedron form a hyperbolic group, then so do cevians to the isogonal conjugates of those centers.*

Proof. This is because in areal coordinates, the isogonal conjugate of a center (x, y, z) is $(a^{-2}/x, b^{-2}/y, c^{-2}/z)$. Thus the result follows from the previous two theorems. \square

Theorem 12.8. *Suppose the edges of a tetrahedron satisfy the condition $h(a_i) + h(b_i) = t$, $i = 1, 2, 3$, for some function $h(x)$ and some constant, t . Then the cevians to the face centers with center function $[h(b) + h(c) - h(a)]^a$ form a hyperbolic group.*

Proof. The hyperbolic condition becomes

$$\begin{aligned} [h(a_1) + h(b_3) - h(b_2)]^a [h(b_1) + h(a_2) - h(b_3)] [h(a_3) + h(b_2) - h(b_1)]^a = \\ [h(b_2) + h(a_1) - h(b_3)]^a [h(a_2) + h(b_3) - h(b_1)] [h(b_1) + h(a_3) - h(b_2)]^a \end{aligned}$$

which immediately follows from

$$\begin{aligned} [h(a_1) + h(b_3) + h(a_2) - t]^a [h(b_1) + h(a_2) + h(a_3) - t]^a [h(a_3) + h(b_2) + h(a_1) - t]^a = \\ [h(b_2) + h(a_1) + h(a_3) - t]^a [h(a_2) + h(b_3) + h(a_1) - t]^a [h(b_1) + h(a_3) + h(a_2) - t]^a \end{aligned}$$

which is identically true. \square

Corollary 12.9. *In a circumscribable tetrahedron, the cevians to the face centers with center function $a^r(b + c - a)$ form a hyperbolic group. This includes the Gergonne point, the Nagel point, the Mittenpunkt, the X_{41} point, and their isogonal and isotomic conjugates.*

Corollary 12.10. *In an orthocentric tetrahedron, the cevians to the face centers with center function $a^r(b^2 + c^2 - a^2)$ form a hyperbolic group. This includes the orthocenter, circumcenter, the crucial point, the X_{25} point, the X_{48} point, and their isogonal and isotomic conjugates.*

Corollary 12.11. *In a harmonic tetrahedron, the cevians to the face centers with center function $a^r(1/b+1/c-1/a)$ form a hyperbolic group. This includes the X_{43} point, the X_{102} point, the X_{117} point, and their isogonal and isotomic conjugates.*

Theorem 12.12. *The power points are the only triangle centers with the property that in any tetrahedron, the cevians to these face centers form a hyperbolic group.*

Proof. Let $F(a, b, c) = af(a, b, c)$ be a hyperbolic center function. Then F must satisfy the functional equation

$$F(b_2, a_1, b_3)F(b_3, b_1, a_2)F(b_1, a_3, b_2) = F(b_3, b_2, a_1)F(b_1, a_2, b_3)F(b_2, b_1, a_3).$$

This equation must be true for all values of $a_1, a_2, a_3, b_1, b_2,$ and b_3 . Rewriting this as

$$(6) \quad \frac{F(b_2, a_1, b_3)}{F(b_3, b_2, a_1)} = \frac{F(b_1, a_2, b_3)}{F(b_3, b_1, a_2)} \cdot \frac{F(b_2, b_1, a_3)}{F(b_1, a_3, b_2)}$$

shows that $F(b_2, a_1, b_3)/F(b_3, b_2, a_1)$ is independent of a_1 , so we may define

$$(7) \quad G(b_2, b_3) = \frac{F(b_2, a_1, b_3)}{F(b_3, b_2, a_1)}.$$

Substituting this in equation (6) yields

$$G(b_2, b_3) = \frac{G(b_1, b_3)}{G(b_1, b_2)}$$

for all $b_1, b_2,$ and b_3 . If we then let

$$H(z) = G(b_1, z)$$

we find that

$$G(x, y) = \frac{H(y)}{H(x)}$$

for all x and y . From the definition of G , (equation 7), we get

$$\frac{H(y)}{H(x)} = \frac{F(x, z, y)}{F(y, x, z)}$$

for all $x, y,$ and z . Since F is homogeneous, this implies that

$$\frac{H(ty)}{H(tx)} = \frac{H(y)}{H(x)}$$

for all $x, y,$ and t . Letting $K(y) = H(y)/H(x)$ gives

$$\frac{K(ty)}{K(y)} = \frac{H(ty)}{H(y)} = \frac{H(tx)}{H(x)} = K(tx).$$

Letting $x = 1$ shows that

$$K(ty) = K(t)K(y)$$

for all t and y , so by the Power Lemma we have $K(x) = x^r$ for some constant r . Denoting $H(1)$ by c , we have

$$H(x) = cK(x) = cx^r.$$

The coordinates for the center are

$$\begin{aligned} F(x, y, z) : F(y, z, x) : F(z, x, y) &= \frac{F(x, y, z)}{F(x, y, z)} : \frac{F(y, z, x)}{F(x, y, z)} : \frac{F(z, x, y)}{F(x, y, z)} \\ &= 1 : \frac{H(x)}{H(y)} : \frac{H(x)}{H(z)} \\ &= \frac{1}{H(x)} : \frac{1}{H(y)} : \frac{1}{H(z)} \\ &= x^r : y^r : z^r \end{aligned}$$

and thus the center is a power point. \square

Theorem 12.13. *In an isodynamic tetrahedron, the center function $a^r g(b, c)$ is hyperbolic for any symmetric homogeneous function g .*

Proof. The hyperbolic condition becomes

$$b_2^r g(a_1, b_3) b_3^r g(b_1, a_2) b_1^r g(a_3, b_2) = b_3^r g(b_2, a_1) b_1^r g(a_2, b_3) b_2^r g(b_1, a_3).$$

Since the tetrahedron is isodynamic, this is equivalent to

$$b_2^r g(a_1, t/a_3) b_3^r g(t/a_1, a_2) b_1^r g(a_3, t/a_2) = b_3^r g(t/a_2, a_1) b_1^r g(a_2, t/a_3) b_2^r g(t/a_1, a_3).$$

Using the fact that g is homogeneous, we see that this is equivalent to

$$b_2^r g(a_1 a_3, t) b_3^r g(t, a_1 a_2) b_1^r g(a_2 a_3, t) = b_3^r g(t, a_1 a_2) b_1^r g(a_2 a_3, t) b_2^r g(t, a_1 a_3)$$

which is easily seen to be an identity since g is symmetric. \square

Corollary 12.14. *In an isodynamic tetrahedron, the cevians to the face centers with center function $a^r (b^q + c^q)^{\pm 1}$ form a hyperbolic group. This includes the Spieker center, the Brocard midpoint, the $X37$, $X38$, and $X42$ points, and their isogonal and isotomic conjugates.*

Conjecture 12.15. *The center functions $a^r g(b, c)$, where g is a symmetric homogeneous function, and r is arbitrary, are the only hyperbolic center functions for an isodynamic tetrahedron.*

13. GENERAL PLANARITY RESULTS

Theorem 13.1. *No triangle center has the property that in any isosceles tetrahedron, the corresponding face centers are coplanar.*

Proof. The algebraic condition for planarity (found by computer) is one of

$$(8) \quad F(a_1, a_2, a_3) + F(a_2, a_3, a_1) = F(a_3, a_1, a_2)$$

$$(9) \quad F(a_2, a_3, a_1) + F(a_3, a_1, a_2) = F(a_1, a_2, a_3)$$

$$(10) \quad F(a_3, a_1, a_2) + F(a_1, a_2, a_3) = F(a_2, a_3, a_1)$$

$$(11) \quad F(a_1, a_2, a_3) + F(a_2, a_3, a_1) + F(a_3, a_1, a_2) = 0.$$

If condition (11) holds, then the center (α, β, γ) in areal coordinates would satisfy $\alpha + \beta + \gamma = 0$, a contradiction.

If condition (8) holds, then since it must be valid for all variables a_1, a_2, a_3 , we see that conditions (9) and (10) would also hold. Adding these three equations yields condition (11) which we have already seen yields a contradiction.

The same contradiction is reached if we assume (9) or (10) holds. \square

Corollary 13.2. *No triangle center has the property that in any tetrahedron, the corresponding face centers are coplanar.*

The following condition was found by our computer program.

Theorem 13.3. *The Feuerbach points on the faces of a tetrahedron are coplanar if the edges of the tetrahedron satisfy the following condition:*

$$\begin{vmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ a_1 b_1 & a_2 b_2 & a_3 b_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

In particular, the Feuerbach points are coplanar for circumscribable, isodynamic, and harmonic tetrahedra. The Feuerbach points would also be coplanar for tetrahedra that satisfy other simple relationships, for example, ones in which $a_i b_i + a_i + b_i$ were constant.

14. GENERAL RESULTS ABOUT CONCURRENT NORMALS

Theorem 14.1. *Let P_1 and P_2 be any two points on faces 1 and 2 of tetrahedron $A_1 A_2 A_3 A_4$, respectively. If normals to the faces at P_1 and P_2 concur, then*

$$(P_2 A_3)^2 + (P_1 A_4)^2 = (P_2 A_4)^2 + (P_1 A_3)^2.$$

Proof. From right triangles $PP_2 A_3$, $PP_1 A_3$, $PP_2 A_4$, and $PP_1 A_4$, we have

$$(A_3 P_2)^2 + (PP_2)^2 = (A_3 P)^2 = (A_3 P_1)^2 + (PP_1)^2$$

and

$$(A_4 P_2)^2 + (PP_2)^2 = (A_4 P)^2 = (A_4 P_1)^2 + (PP_1)^2.$$

Subtracting one equation from the other gives us the desired result. \square

The converse is also true.

Theorem 14.2 (Tabov [14]). *Let P_1 and P_2 be any two points on faces 1 and 2 of tetrahedron $A_1 A_2 A_3 A_4$, respectively. If*

$$(P_2 A_3)^2 + (P_1 A_4)^2 = (P_2 A_4)^2 + (P_1 A_3)^2,$$

then normals to the faces at P_1 and P_2 concur.

15. MISCELLANEOUS CONJECTURES

The following conjectures are backed up by the data, but I don't have formal proofs.

Conjecture 15.1. *If the central tetrahedron is isosceles, then the reference tetrahedron is isosceles.*

Conjecture 15.2. *If the central tetrahedron is regular, then the reference tetrahedron is regular.*

Conjecture 15.3. *If the central tetrahedron is similar to the reference tetrahedron, then the center must be the centroid.*

Conjecture 15.4. *If the cevians to the corresponding face centers have the same length, then the tetrahedron is isosceles.*

APPENDIX A. CENTER FUNCTIONS AND TRILINEAR COORDINATES

In this appendix, we give the basic information about center functions and trilinear coordinates that the reader needs to know.

Before giving the definition of a triangle center, let us review the concept of trilinear coordinates. If ABC is a fixed reference triangle in the plane (with sides of lengths a , b , and c), and if P is an arbitrary point in the plane, then the trilinear coordinates of P are (α, β, γ) where α , β , γ are the signed distances from P to the sides BC , CA , AB , respectively. The three coordinates satisfy the condition

$$(12) \quad a\alpha + b\beta + c\gamma = 2K$$

where K is the area of $\triangle ABC$. If α , β , and γ are any three real numbers (with $a\alpha + b\beta + c\gamma \neq 0$), then there is a unique point P in the plane whose trilinear coordinates are proportional to $\alpha : \beta : \gamma$. Thus (α, β, γ) may be considered to be the trilinear coordinates of P even if condition (12) is not satisfied. If condition (1) is satisfied, then we refer to the coordinates as exact trilinear coordinates.

A *center function* is a nonzero function $f(a, b, c)$ that is homogeneous in a , b , and c and symmetric in b and c . In other words, a center function must satisfy the following two conditions for all a, b, c, t , and some integer r :

$$(C1) \quad f(ta, tb, tc) = t^r f(a, b, c)$$

$$(C2) \quad f(a, c, b) = f(a, b, c)$$

A *center* is an ordered triple $\alpha : \beta : \gamma$ given by

$$\alpha = f(a, b, c), \quad \beta = f(b, c, a), \quad \gamma = f(c, a, b)$$

for some center function $f(a, b, c)$. See [10] for more details. If f is a polynomial, then the center is called a polynomial center.

If $P = (\alpha, \beta, \gamma)$, then the point $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$ is denoted by P^{-1} and is called the isogonal conjugate of P . For the geometric meaning of isogonal conjugates, consult [1].

More information about center functions and trilinear coordinates can be found in [10], [11], and [12].

APPENDIX B. AREAL COORDINATES

In this appendix, we give the basic information about areal coordinates that the reader needs to know.

Let ABC be a fixed reference triangle in the plane. If P is an arbitrary point in the plane of $\triangle ABC$, then the areal coordinates of P are given by (x, y, z) where

$$\begin{aligned} x &= [PBC]/[ABC], \\ y &= [PCA]/[ABC], \\ z &= [PAB]/[ABC]. \end{aligned}$$

The three areal coordinates satisfy the condition

$$(13) \quad x + y + z = 1.$$

Areal coordinates are also known as barycentric coordinates.

If x , y , and z are any three real numbers (with $x + y + z \neq 0$), then there is a unique point P in the plane whose areal coordinates are proportional to $x : y : z$. Thus (x, y, z) may be considered to be the areal coordinates of P even if condition (13) is not satisfied. The three coordinates are proportional to the areas formed by P and the sides of the reference triangle ABC . If condition (13) is satisfied, then we refer to the coordinates as exact areal coordinates.

Areal coordinates, (x, y, z) , can be transformed to trilinear coordinates, (α, β, γ) , and vice versa, by the following formulas:

$$\alpha = x/a, \quad \beta = y/b, \quad \gamma = z/c$$

where a , b , and c are the lengths of the sides of the reference triangle.

If $P = (x, y, z)$, then the point (x^{-1}, y^{-1}, z^{-1}) is denoted by P^T and is called the isotomic conjugate of P . For the geometric meaning of isotomic conjugates, consult [1]. In terms of trilinear coordinates, the isotomic conjugate of (α, β, γ) is $(\alpha^{-1}/a^2, \beta^{-1}/b^2, \gamma^{-1}/c^2)$.

More information about areal coordinates can be found in [3] and [9].

APPENDIX C. TETRAHEDRAL COORDINATES

In this appendix, we give the basic information about tetrahedral coordinates that the reader needs to know.

The 3-dimensional analog of areal coordinates are tetrahedral coordinates. Let $[T]$ denote the volume of a tetrahedron T . Let $A_1A_2A_3A_4$ be a fixed reference tetrahedron in space with edges of lengths $A_2A_3 = a_1$, $A_3A_1 = a_2$, $A_1A_2 = a_3$, $A_1A_4 = b_1$, $A_2A_4 = b_2$, and $A_3A_4 = b_3$, so that the edges of lengths a_i and b_i are opposite each other, $i = 1, 2, 3$.

If P is an arbitrary point in space, then the tetrahedral coordinates of P are given by (x_1, x_2, x_3, x_4) where

$$\begin{aligned} x_1 &= [PA_2A_3A_4]/V, \\ x_2 &= [PA_1A_3A_4]/V, \\ x_3 &= [PA_1A_2A_4]/V, \\ x_4 &= [PA_2A_3A_4]/V, \end{aligned}$$

where $V = [A_1A_2A_3A_4]$ is the volume of the reference tetrahedron. The four tetrahedral coordinates satisfy the condition

$$(14) \quad x_1 + x_2 + x_3 + x_4 = 1.$$

If x_i , $i = 1, 2, 3, 4$ are any four real numbers (with nonzero sum), then there is a unique point P in space whose tetrahedral coordinates are proportional to $x_1 : x_2 : x_3 : x_4$. Thus (x_1, x_2, x_3, x_4) may be considered to be the tetrahedral coordinates of P even if condition (14) is not satisfied. The four coordinates are proportional to the volumes formed by P and the faces of the reference tetrahedron $ABCD$. If condition (14) is satisfied, then we refer to the coordinates as exact tetrahedral coordinates.

For more information about tetrahedral coordinates, see [4] and [7].

APPENDIX D. FORMULAS

In this appendix, we collect together formulas about tetrahedral coordinates that are needed in this paper. All points are given using exact tetrahedral coordinates.

POINTS

Formula 1: Coordinates of a point ([7, p. 65]):

$$(x, y, z, w)$$

where $x + y + z + w = 1$. The coordinates are proportional to the volumes of the tetrahedra formed by the point and the faces of the reference tetrahedron. If a more symmetric notation is needed, we will use the alternate form (X_1, X_2, X_3, X_4) .

Formula 2: Condition for 4 points (x_i, y_i, z_i, w_i) , $i = 1, 2, 3, 4$ to be coplanar:

$$\begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \\ x_4 & y_4 & z_4 & w_4 \end{vmatrix} = 0$$

or equivalently,

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0.$$

Formula 3: Condition for 3 points (x_i, y_i, z_i, w_i) , $i = 1, 2, 3$ to be collinear:

$$\frac{x_2 - x_1}{x_3 - x_1} = \frac{y_2 - y_1}{y_3 - y_1} = \frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1}.$$

Formula 4: Square of distance between points (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) ([7, p. 66]):

$$(y_1 - y_2)(z_2 - z_1)a_1^2 + (z_1 - z_2)(x_2 - x_1)a_2^2 + (x_1 - x_2)(y_2 - y_1)a_3^2 + \\ (x_1 - x_2)(w_2 - w_1)b_1^2 + (y_1 - y_2)(w_2 - w_1)b_2^2 + (z_1 - z_2)(w_2 - w_1)b_3^2.$$

This can be written in the symmetrical form:

$$- \sum_{i,j} d_{i,j}^2 (X_i - X'_i)(X_j - X'_j)$$

representing the square of the distance from (X_1, X_2, X_3, X_4) to (X'_1, X'_2, X'_3, X'_4) . In this symmetrical notation, the summation symbol

$$\sum_{i,j} \quad \text{means} \quad \sum_{\substack{i,j=1 \\ i < j}}^4$$

and $d_{i,j}$ represents the length of edge $A_i A_j$, so that

$$d_{2,3} = a_1, \quad d_{1,3} = a_2, \quad d_{1,2} = a_3, \quad d_{1,4} = b_1, \quad d_{2,4} = b_2, \quad d_{3,4} = b_3.$$

LINES

Formula 5: General equation of a straight line ([7, p. 67]):

$$\frac{x - x_0}{K} = \frac{y - y_0}{L} = \frac{z - z_0}{M} = \frac{w - w_0}{N}$$

where $K + L + M + N = 0$. The line passes through the point (x_0, y_0, z_0, w_0) . The quadruple, (K, L, M, N) , represents the direction of the line and is called the direction vector. Two lines are parallel if and only if they have the same direction vector (or a multiple thereof). Note that some of K, L, M, N may be 0 because the condition $(x - x_0)/K = (y - y_0)/L$ is really an abbreviation for $(x - x_0)L = (y - y_0)K$ which does not involve any possible divisions by 0.

Formula 6: Parametric equation of a straight line:

$$(x_0 + Kt, y_0 + Lt, z_0 + Mt, w_0 + Nt).$$

This formula yields all the points on the line through (x_0, y_0, z_0, w_0) with direction vector (K, L, M, N) as t varies through the real numbers.

Formula 7: Equation of line through (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) :

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \frac{w - w_1}{w_2 - w_1}.$$

Formula 8: Coordinates of point that divides the line joining points (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) in the ratio $\mu : \lambda$ ([7, p. 65]):

$$\left(\frac{\lambda x_1 + \mu x_2}{\lambda + \mu}, \frac{\lambda y_1 + \mu y_2}{\lambda + \mu}, \frac{\lambda z_1 + \mu z_2}{\lambda + \mu}, \frac{\lambda w_1 + \mu w_2}{\lambda + \mu} \right).$$

Formula 9: Condition for two lines $(x - x_i)/K_i = (y - y_i)/L_i = (z - z_i)/M_i = (w - w_i)/N_i$, $i = 1, 2$ to be parallel:

$$K_1 : L_1 : M_1 : N_1 = K_2 : L_2 : M_2 : N_2$$

or equivalently

$$\frac{K_1}{K_2} = \frac{L_1}{L_2} = \frac{M_1}{M_2} = \frac{N_1}{N_2}.$$

Formula 10: Condition for two lines $(x - x_i)/K_i = (y - y_i)/L_i = (z - z_i)/M_i = (w - w_i)/N_i$, $i = 1, 2$ to be perpendicular ([7, p. 68]):

$$a_1^2(L_1M_2 + L_2M_1) + a_2^2(K_1M_2 + K_2M_1) + a_3^2(K_1L_2 + K_2L_1) + b_1^2(K_1N_2 + K_2N_1) + b_2^2(L_1N_2 + L_2N_1) + b_3^2(M_1N_2 + M_2N_1) = 0.$$

Formula 11: Condition for two lines $(x - x_i)/K_i = (y - y_i)/L_i = (z - z_i)/M_i = (w - w_i)/N_i$, $i = 1, 2$ to intersect (or be parallel):

$$\begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ K_1 & L_1 & M_1 & N_1 \\ x_2 & y_2 & z_2 & w_2 \\ K_2 & L_2 & M_2 & N_2 \end{vmatrix} = 0$$

or equivalently

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ K_1 & L_1 & M_1 \\ K_2 & L_2 & M_2 \end{vmatrix} = 0.$$

Formula 12: Direction cosines of the line $(x - x_0)/K = (y - y_0)/L = (z - z_0)/M = (w - w_0)/N$:

$$\frac{KF_1}{3V\sigma}, \frac{LF_2}{3V\sigma}, \frac{MF_3}{3V\sigma}, \frac{NF_4}{3V\sigma},$$

where V is the volume of the reference tetrahedron, F_i is the area of face i (the face opposite vertex A_i), and σ is determined from

$$a_1^2 LM + a_2^2 MK + a_3^2 KL + b_1^2 KN + b_2^2 LN + b_3^2 MN = -\sigma^2.$$

The direction cosines are the cosines of the angles that the line makes with the normals to the four faces of the reference tetrahedron. They are proportional to the direction vector.

PLANES

Formula 13: General equation of a plane ([7, p. 69]):

$$Ax + By + Cz + Dw = 0$$

where not all coefficients are 0. The coefficients A, B, C, D , are proportional to the directed distances from the plane to the vertices of the reference tetrahedron.

Formula 14: Equation of the plane through 3 points, (x_i, y_i, z_i, w_i) , $i = 1, 2, 3$ ([7]):

$$\begin{vmatrix} x & y & z & w \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{vmatrix} = 0.$$

Formula 15: Condition for planes $A_1x + B_1y + C_1z + D_1w = 0$ and $A_2x + B_2y + C_2z + D_2w = 0$ to be parallel ([7, p. 70]):

$$\frac{A_1 - D_1}{A_2 - D_2} = \frac{B_1 - D_1}{B_2 - D_2} = \frac{C_1 - D_1}{C_2 - D_2}.$$

Formula 16: Equation of the plane through the point (x_1, y_1, z_1, w_1) and the line $(x - x_2)/K = (y - y_2)/L = (z - z_2)/M = (w - w_2)/N$:

$$\begin{vmatrix} x & y & z & w \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ K & L & M & N \end{vmatrix} = 0.$$

Formula 17: Equation of plane through the line $(x - x_1)/K_1 = (y - y_1)/L_1 = (z - z_1)/M_1 = (w - w_1)/N_1$ and parallel to the line with direction vector (K_2, L_2, M_2, N_2) :

$$\begin{vmatrix} x & y & z & w \\ x_1 & y_1 & z_1 & w_1 \\ K_1 & L_1 & M_1 & N_1 \\ K_2 & L_2 & M_2 & N_2 \end{vmatrix} = 0.$$

Formula 18: Point of intersection of the line $(x - x_1)/K = (y - y_1)/L = (z - z_1)/M = (w - w_1)/N$ and the plane $Ax + By + Cz + Dw = 0$:

$$(x_1 - rK, y_1 - rL, z_1 - rM, w_1 - rN)$$

where

$$r = \frac{Ax_1 + By_1 + Cz_1 + Dw_1}{AK + BL + CM + DN}.$$

Formula 19: Condition for the line $(x - x_1)/K = (y - y_1)/L = (z - z_1)/M = (w - w_1)/N$ to be parallel to the plane $Ax + By + Cz + Dw = 0$:

$$AK + BL + CM + DN = 0.$$

APPENDIX E. TETRAHEDRON CENTERS

In this appendix, we collect together information about various “centers” associated with a tetrahedron. We give the tetrahedral coordinates for the more well-known such centers and explain why some of these centers were not included in our study.

CENTROID

The centroid, G , of a tetrahedron ([2, p. 54]) is the center of gravity of unit masses placed on the vertices. Thus it has barycentric coordinates of

$$G = (1, 1, 1, 1).$$

The exact tetrahedral coordinates are $(1/4, 1/4, 1/4, 1/4)$. The centroid is also the intersection of the medians of the tetrahedron (the lines from a vertex to the centroid of the opposite face).

INCENTER

The incenter, I , of a tetrahedron ([2, p. 76]) is the center of the sphere inscribed in the tetrahedron (touching each of the faces internally). If we let r be the inradius of the tetrahedron, then the volume of $IA_2A_3A_4$ is $\frac{1}{3}rF_1$. Similarly for the other three volumes formed by I and the faces of the tetrahedron. These four volumes sum to V , the volume of the tetrahedron. Thus $r = 3V/(F_1 + F_2 + F_3 + F_4)$. The incenter is equidistant from each face of the tetrahedron. Thus, the tetrahedral coordinates are

$$I = (F_1, F_2, F_3, F_4).$$

To convert to exact tetrahedral coordinates, each coordinate should be divided by F , the surface area of the tetrahedron.

CIRCUMCENTER

The circumcenter, O , of a tetrahedron ([2, p. 56]) is the center of the circumscribed sphere. If (O_x, O_y, O_z, O_w) are the coordinates for the circumcenter of our reference tetrahedron, and if R denotes the circumradius, then we can set up 4 equations in O_x, O_y, O_z, O_w , and R :

$$d(O, A_1)^2 = R^2$$

$$d(O, A_2)^2 = R^2$$

$$d(O, A_3)^2 = R^2$$

$$d(O, A_4)^2 = R^2$$

where $d(P_1, P_2)$ denotes the distance between points P_1 and P_2 . This distance formula is given by formula 4. Upon subtracting equation 2 from equation 1, equation 3 from equation 2, and equation 4 from equation 3, we get 3 linear equations in O_x, O_y, O_z , and O_w . Combining these with $O_x + O_y + O_z + O_w = 1$ gives us 4 linear equations in 4 unknowns. These are straightforward to solve. The value of O_x found is proportional to

$$O_x = a_1^2 b_1^2 (b_2^2 + b_3^2 - a_1^2) + a_2^2 b_2^2 (b_3^2 + a_1^2 - b_2^2) + a_3^2 b_3^2 (a_1^2 + b_2^2 - b_3^2) - 2a_1^2 b_2^2 b_3^2.$$

The values of O_y , O_z , and O_w are similar and can be obtained from O_x by the mappings given in display (5). Specifically,

$$\begin{aligned} O_y &= a_1^2 b_1^2 (a_2^2 + b_3^2 - b_1^2) + a_2^2 b_2^2 (b_3^2 + b_1^2 - a_2^2) + a_3^2 b_3^2 (b_1^2 + a_2^2 - b_3^2) - 2b_1^2 a_2^2 b_3^2, \\ O_z &= a_1^2 b_1^2 (b_2^2 + a_3^2 - b_1^2) + a_2^2 b_2^2 (a_3^2 + b_1^2 - b_2^2) + a_3^2 b_3^2 (b_1^2 + b_2^2 - a_3^2) - 2b_1^2 b_2^2 a_3^2, \\ O_w &= a_1^2 b_1^2 (a_2^2 + a_3^2 - a_1^2) + a_2^2 b_2^2 (a_3^2 + a_1^2 - a_2^2) + a_3^2 b_3^2 (a_1^2 + a_2^2 - a_3^2) - 2a_1^2 a_2^2 a_3^2. \end{aligned}$$

MONGE POINT

The Monge point, M , of a tetrahedron ([2, p. 76]) is the common intersection point of the six planes through the midpoints of the edges of the tetrahedron and perpendicular to the opposite edges. The Monge point is the symmetric of the circumcenter with respect to the centroid ([2, p. 77]) and thus its coordinates can be found from them:

$$M = 2G - O.$$

Definition 7. The three points G , O , and M lie on a straight line called the *Euler line* of the tetrahedron ([2, p. 77]).

EULER POINT

The Euler point, E , corresponds to the nine-point center in the plane. It is frequently called the twelve point center ([2, p. 289]) because it is the center of a sphere that passes through 12 notable points in the tetrahedron. The Euler point lies on the Euler line of the tetrahedron and divides the segment MO in the ratio 1 : 2 and it divides the segment GM in the ratio 1 : 2. Thus its coordinates can be found from the coordinates of those points:

$$E = (2G + M)/3.$$

ORTHOCENTER

The altitudes of a tetrahedron do not normally intersect. They intersect if and only if the tetrahedron is orthocentric and in that case, the intersection point (the orthocenter) coincides with the Monge point of the tetrahedron ([2, p. 71]). We do not include the orthocenter of a tetrahedron as a distinguished point in our study since it is not present in all tetrahedra.

SYMMEDIAN POINT (Lemoine Point)

In a tetrahedron, the cevians to the symmedian points on the opposite faces do not normally intersect. They intersect if and only if the tetrahedron is isodynamic ([2, p. 315]). The point of intersection is called the symmedian point of the tetrahedron. We do not include the symmedian point of a tetrahedron as a distinguished point in our study since it is not present in all tetrahedra.

GERGONNE POINT

In a tetrahedron, the cevians to the Gergonne points on the opposite faces do not normally intersect. They intersect if and only if the tetrahedron is circumscribable ([2, p. 299]). The point of intersection is called the Gergonne point of the tetrahedron. We do not include the Gergonne point of a tetrahedron as a distinguished point in our study since it is not present in all tetrahedra.

NAGEL POINT

In a tetrahedron, the cevians to the Nagel points on the opposite faces do not normally intersect. They intersect if and only if the tetrahedron is circumscribable

([2, p. 299]). The point of intersection is called the Nagel point of the tetrahedron. We do not include the Nagel point of a tetrahedron as a distinguished point in our study since it is not present in all tetrahedra.

FERMAT POINT

In a tetrahedron, the cevians to the points of tangency of the opposite faces with the insphere do not normally intersect. If they intersect, the tetrahedron is called an isogonic tetrahedron ([2, p. 328]). The point of intersection is called the Fermat point of the tetrahedron. In this case, the insphere touches the faces at the Fermat point of each face. We do not include the Fermat point of a tetrahedron as a distinguished point in our study since it is not present in all tetrahedra.

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