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A Purely Synthetic Proof of the Dao's Eight Circles Theorem

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Abstract. In this note we present a simple elementary proof of Dao's theorem on eight circles.

Keywords. Miquel's six circles theorem, the bundle theorem, Pascal's theorem, Brianchon's theorem, Carnot' theorem, Dao's theorem

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1. INTRODUCTION

In 2013, Dao Thanh Oai presented a remarkable theorem on eight circles:

Theorem 1.1. *(Dao-[1]). Let the six points $A_1, A_2, \cdots, A_6$ lie in that order on a circle, and the six points $B_1, B_2, \cdots, B_6$ lie in that order on another circle. If the quadruples $A_i, A_{i+1}, B_{i+1}, B_i$ lie on circles with centres Oi for $i = 1, 2, \cdots, 5$, then prove that A_1, B_1, B_6, A_6 must be also lie on a circle. Furthermore if O_6 is the centre of the new circle, then lines O_1O_4, O_2O_5 and O_3O_6 are concurrent.*

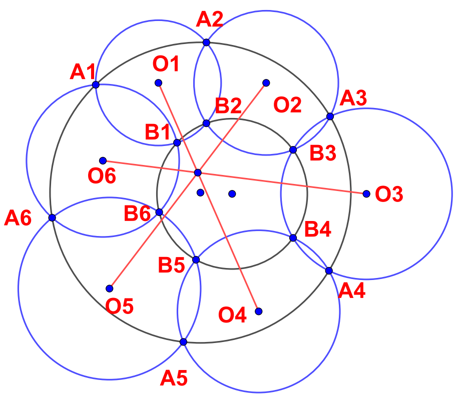

FIGURE 1.

In order to solve Theorem 1.1, we can directly use J. C. Fisher's Lemma. You can see the proof of Fisher's Lemma by M. Bataille in [2][3]. In June 2016, T. O. Dao presented this theorem and its dual again in his article [4].

At the beginning of 2018, G. Ge'vay published a paper on the extension of Miquel's theorem for an even number of circles [5]. A'kos G. Horva'th also presented an analysis of a similar problem regarding the class of conic curves, using more advanced mathematical tools [6]. You can see the dual theorem and some more properties of the theorem in [8], [9], [10], [11].

In this note we present a simple proof of Dao's theorem on eight circles using two simple tools the Lemma of orthodiagonal quadrilateral and the Carnot's three perpendicular lines theorem. Those terms and facts of Euclidean geometry – since they are generally known and simple, I refrained from presenting their proofs.

2. A PROOF OF THEOREM 1.1

Lemma 2.1. ([7]). *For any 4 points T, X, Y, Z on the plane:*

$$TX \perp YZ \Leftrightarrow TY^2 - TZ^2 = XY^2 - XZ^2$$

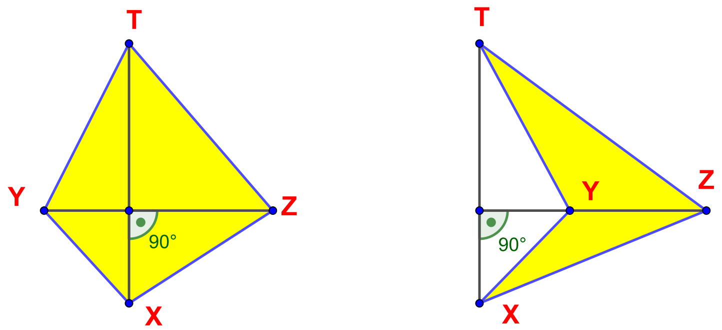

FIGURE 2.

Theorem 2.1. *(Carnot-[8]). Let there be given a triangle DEF and points L, M, N. Lines d_D, d_E, d_F go in order through L, M, N, and are in the same order perpendicular to EF, FD, DE. In this situation, d_D, d_E, d_F are concurrent if and only if:*

$$(LE^2 - LF^2) + (MF^2 - MD^2) + (ND^2 - NE^2) = 0$$

or $f(L, M, N, DEF) = 0$ where we shall call $f(L, M, N, DEF) = (LE^2 - LF^2) + (MF^2 - MD^2) + (ND^2 - NE^2)$ Carnot's value in relations to points L, M, N and triangle DEF.

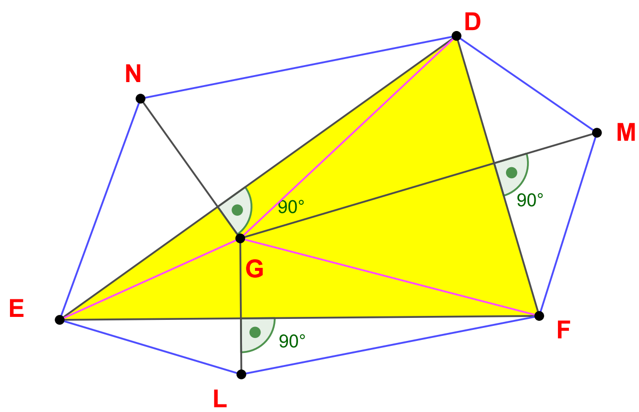

FIGURE 3.

The first thesis - that A_1, B_1, B_6, A_6 must lie on a one circle – is easy to prove by double use of the Miquel's six circle theorem. Applying Miquel's theorem to the circles (O), (I), $(A_1B_1B_2A_2)$, $(A_2B_2B_3A_3)$, $(A_3B_3B_4A_4)$ we reach a conclusion that A_1, B_1, B_4, A_4 lie on a circle. Applying the same theorem to the circles (O), (I), $(A_1B_1B_4A_4)$, $(A_4B_4B_5A_5)$, $(A_5B_5B_6A_6)$ we reach a conclusion that A_1, B_1, B_6, A_6 also lie on the circle.

Now we will prove that the lines O_1O_4, O_2O_5 and O_3O_6 are concurrent

Proof. (See Figure 4 or Figure 5). Let's mark intersection points of pairs of lines A_1B_1 & A_4B_4, A_2B_2 & A_5B_5 and A_3B_3 & A_6B_6 as P, Q, R and name radiuses of circles (O_1), (O_2), (O_3), (O_4), (O_5) and (O_6) as r_1, r_2, r_3, r_4, r_5 and r_6. As we've noted before, quad-

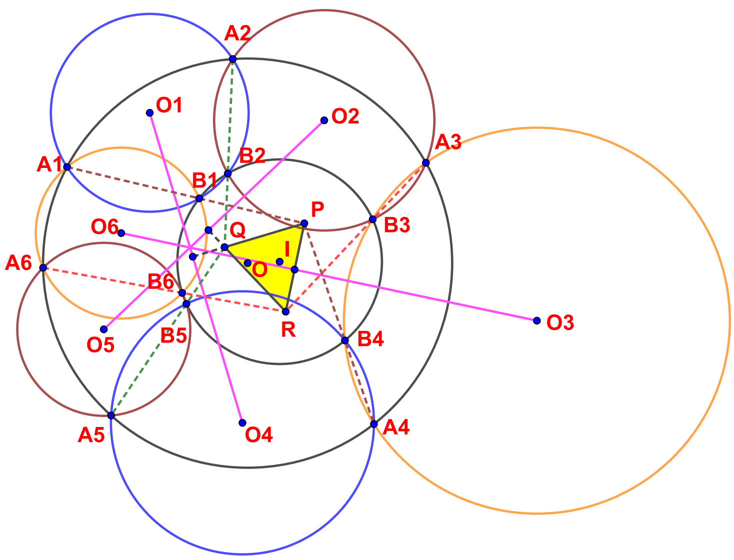

FIGURE 4.

rangle $A_1B_1B_4A_4$ is circumscribed, and for the same reasons quadrangles $A_2B_2B_5A_5$, $A_3B_3B_6A_6$ are also circumscribed. Therefore:

- P is the radical centre of five circles $(A_1B_1A_4B_4)$, (O_6), (O_1), (O_3) and (O_4) (1)

- Q is the radical centre of five circles $(A_2B_2A_5B_5)$, (O_1), (O_2), (O_4) and (O_5) (2)

- R is the radical centre of five circles $(A_3B_3A_6B_6)$, (O_2), (O_3), (O_5) and (O_6) (3)

Therefore:

- From (1) and (2) we assume that PQ is a radical axis of circles (O_1) and (O_4), therefore $PQ \perp O_1O_4$. Using Lemma 2.1, we receive an equation:

$O_1P^2 - O_1Q^2 = O_4P^2 - O_4Q^2$ (4)

- From (2) and (3) we assume that QR is a radical axis of circles (O_2) and (O_5), therefore $QR \perp O_2O_5$. Using Lemma 2.1, we receive an equation:

$O_2Q^2 - O_2R^2 = O_5Q^2 - O_5R^2$ (5)

- From (1) and (3) we assume that RP is a radical axis of circles (O_3) and (O_6), therefore $RP \perp O_3O_6$. Using Lemma 2.1, we receive an equation:

$O_3R^2 - O_3P^2 = O_6R^2 - O_6P^2$ (6)

Carnot's value of points O_1, O_2, O_3 in relation to triangle PQR equals:

$$f(O_1,O_2,O_3,PQR) = (O_1P^2 - O_1Q^2) + (O_2Q^2 - O_2R^2) + (O_3R^2 - O_3P^2)$$

According to equations (4), (5) and (6), we also have:

$$f(O_1,O_2,O_3,PQR) = (O_4P^2 - O_4Q^2) + (O_5Q^2 - O_5R^2) + (O_6R^2 - O_6P^2)$$

Therefore, by adding sides we receive:

$2f(O_1,O_2,O_3,PQR) = (O_1P^2-O_1Q^2)+(O_2Q^2-O_2R^2)+(O_3R^2-O_3P^2)+(O_4P^2-O_4Q^2)+(O_5Q^2-O_5R^2)+(O_6R^2-O_6P^2) = (O_1P^2-O_6P^2)+(O_2Q^2-O_1Q^2)+(O_3R^2-O_2R^2)+(O_4P^2-O_3P^2)+(O_5Q^2-O_4Q^2)+(O_6R^2-O_5R^2)$ (7)

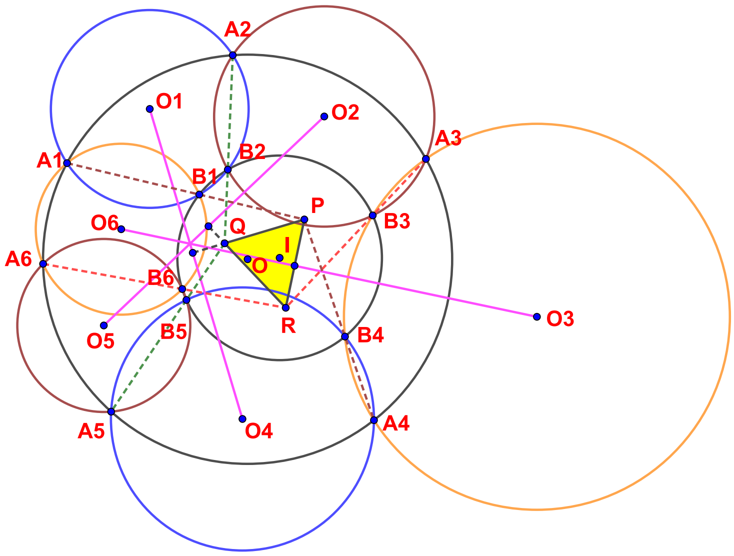

FIGURE 5.

Since the line crossing points of intersection of two circles is always perpendicular to the line connecting their centres, it means that: $PA_1 \perp O_6O_1$, $QA_2 \perp O_1O_2$, $RA_3 \perp O_2O_3$, $PA_4 \perp O_3O_4$, $QA_5 \perp O_4O_5$ and $RA_6 \perp O_5O_6$.

According to Lemma 2.1, we have the following equations:

$$O_1P^2 - O_6P^2 = O_1A_1^2 - O_6A_1^2$$

$$O_2Q^2 - O_1Q^2 = O_2A_2^2 - O_1A_2^2$$

$$O_3R^2 - O_2R^2 = O_3A_3^2 - O_2A_3^2$$

$$O_4P^2 - O_3P^2 = O_4A_4^2 - O_3A_4^2$$

$$O_5Q^2 - O_4Q^2 = O_5A_5^2 - O_4A_5^2$$

$$O_6R^2 - O_5R^2 = O_6A_6^2 - O_5A_6^2$$

Substituting these equations to the equation (7), we receive:

$2f(O_1,O_2,O_3,PQR) = (O_1A_1^2-O_6A_1^2)+(O_2A_2^2-O_1A_2^2)+(O_3A_3^2-O_2A_3^2)+(O_4A_4^2-O_3A_4^2)+(O_5A_5^2-O_4A_5^2)+(O_6A_6^2-O_5A_6^2) = (r_1^2-r_6^2)+(r_2^2-r_1^2)+(r_3^2-r_2^2)+(r_4^2-r_3^2)+(r_5^2-r_4^2)+(r_6^2-r_5^2) = 0$

Which means that $f(O_1,O_2,O_3,PQR) = 0$, therefore, according to the Carnot's theorem O_1O_4, O_2O_5 and O_3O_6 are concurrent. □

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